COMPARISONS AND STABILITY ANALYSIS OF LINEARIZED EQUATIONS OF MOTION FOR A BASIC BICYCLE MODEL

A Thesis

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by

Richard Scott Hand

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ABSTRACT

This thesis focuses on increasing understanding of bicycle self-stability. Assuming that short term rider controlled bicycle stability closely relates to bicycle self-stability, the analysis may be useful in providing general design guidelines.

Because there is no established set of equations of motion agreed upon in the literature, a 'basic' bicycle model having a rigid frame and rider, and rigid knife-edge discs as wheels is introduced and the linearized equations for hands-off motion are derived. When these equations are compared to 20 other systems of equations in the literature, three past sets of equations are found to be in complete agreement with those derived. Other sets of equations were not as general, were missing terms, had sign errors, or disagreed in other ways.

The Routh-Hurwitz stability criteria are then applied to the derived linearized equations to develop seven analytical design criteria. From the design criteria, limiting criteria which bound any interval of velocities in which the bicycle is self-stable are discussed. An analytical proof shows that the 'basic' bicycle can *not* be stable if the steering axis tilt, mechanical trail, and the distance forward from the steering axis of the front assembly center of mass are all zero. Stabilization of the bicycle by these parameters is then discussed and numerical examples given. In addition, it is shown with specific examples that the 'basic' bicycle model with no angular momentum due to the spinning of the wheels can be made stable; and that a bicycle with negative trail can be stable.

A summary of the conclusions of past work in the bicycle/motorcycle handling and stability literature shows that most analyses have only explored small variations from a standard design popular in their day, and have provided few general conclusions or design guidelines. As a method of exploring any design configuration, a simple PC based computer program is described, and illustrated with numerical examples.

Recommendations include using the analytical design criteria in combination with information from eigenvalue-eigenvector studies to verify results of past investigations, and further exploration of new, possibly radically different, bicycle designs.

BIOGRAPHICAL SKETCH

R. Scott Hand was born on August 20, 1963, in Akron, Ohio. His earliest memories are from Toledo, Ohio. There, at the age of 4, he first experienced dynamic vehicle instability; while Bernie "Barn" Zayre pushed him from behind in his red wagon, he careened out of control and into a neighbor's basement window. Little did he know then that someday he would study vehicle dynamics.

Early in 1968, his family moved to Plymouth, Michigan, a suburb of Detroit. Aside from three years in Arlington, Texas, his secondary schooling took place in Plymouth, and he graduated from Plymouth Canton High School on June 18, 1981.

At the age of 17, while listening to Zen Buddist philosopher Alan Watts on the radio, he convinced himself he should attend Worcester Polytechnic Institute because of their liberal philosophy of education.

With his parents support, he entered Worcester in the fall of 1981. He has many fond memories of being a student at Worcester, and it was there that he developed his interest in education and decided to continue on to graduate school with the hope of becoming a professor. He graduated from Worcester on May 31, 1985 with a Bachelor of Science degree specializing mechanical engineering.

In the fall of 1985, he entered graduate school at Cornell University in the Department of Theoretical and Applied Mechanics. He continued his education, gained experience as a teaching assistant, and received exposure to academic research. He found teaching very rewarding and loved to interact with students.

He currently works for Dresser-Rand Company in Painted Post, N. Y. where he is a project engineer and is designing rotating compressors.

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I wish to thank my friends and role models, Professors Jack Boyd and Ray Hagglund of Worcester Polytechnic Institute, for acting as my advisors during my undergraduate education. Without their concern for my future, I might not have attended graduate school.

This thesis would not have been accomplished without the support of my friends, in particular: Les 'Shuffle' Schaeffer, who stayed with me till the end and gave me a great deal of support while running and playing squash; Bob Kolvoord and family, who showed me you can be a graduate student *and* have a family life; The Hudson Hoose, for their love and support, especially: Cesca Verdier, Sarahita Simpkins, Richard Genest, Marcel Hawiger, and Carole Stone (Stoney Woney Doo Dah); and my friends Andrew and Ully who periodically reminded me that graduate school was for my benefit.

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I. INTRODUCTION

Anyone who has ridden a bicycle has had some practical experience in bicycle stability. However, some of us have experienced bicycle instability. Perhaps it was when turning one's head to see if there was an oncoming car, which created an alarming swerve to the middle of the road; riding down a hill when your steering began to uncontrollably shake back and forth; losing momentary control when your bicycle hit a bump in the road; or being unable to ride with no hands on some bicycles. These incidents of bicycle instability could cause serious accidents, which possibly can be avoided through improved bicycle design.

The goal of our research is to improve bicycle design.[1] As a means to achieve that goal the objective of this thesis was to gain understanding about bicycle stability and bicycle instability. The premise on which this thesis is written is that

[1] This thesis was done in cooperation with the Cornell Bicycle Research Project (CBRP).

short term rider controlled stability is closely linked to the bicycle's ***self-stability***, as defined in Chapter V. We believe the premise is valid because in the short term the rider has little control. Hence, design change effects on the bicycle's self-stability will have almost immediate impact on rider's perspective of handling, and possibly performance. Based on this premise, this thesis focuses on understanding the self-stability of the bicycle, but much of what is said and derived may apply to rider controlled stability.

The study of bicycle and motorcycle stability has attracted attention from such well known mechanicians as Rankine [1869], Sommerfeld and Klein [1903], Timoshenko [1948], Den Hartog [1948], and Kane [1975]. Past works on the subject range from purely empirical analyses to nonlinear computer simulation studies. However, no set of equations of motion has been agreed upon. The literature lacks comparisons of previous works. Past work has aimed at optimizing the particular type of bicycle or motorcycle popular in its day, and does not develop understanding of the relations between design parameters for bicycle stability. As a result, no general guidelines or design criteria exist for bicycle builders and riding enthusiasts.

Currently what makes a bicycle more stable, or unstable, is misunderstood. Bicycle design has evolved mostly from experience. How much rider input is required or desired? What do riders sense as being stable or unstable? What is the effect of design changes on stability? These are some questions currently not fully understood from previous research. It is this lack of understanding that has led to some bicycles which are unpleasant, or even dangerous, to ride.

This thesis develops and explores analytical design criteria which can be readily used to gain insight on qualitative and quantitative handling characteristics of different bicycles and riders.

The first step in research for this thesis was to find the correct equations of motion and put them in a simple form. Because in the past so few have compared their work to others, and disagreements existed in the equations presented, the derivation of the equations of motion was required. Chapter III and Appendix A are devoted to this task. The linearized equations of motion for a 'basic' bicycle model are derived using Lagrange's equations with nonholonomic constraints on the wheel contacts with the ground, and resulting generalized forces of constraint.

In order to check our equations to see if they were correct, the resulting equations are compared to published derivations. Three sets of equations are found to agree with ours completely (one author sighted another's equations, so really only two papers derive equations that agree with those presented in Chapter III); all were derived using Newton's Laws. A few other sets agree after slight corrections and/or modifications.

To these the linearized equations we apply the Routh-Hurwitz stability criteria and develop analytical design criteria, which we simplify as much as possible. From these design criteria we prove analytically that the 'basic' bicycle model can *not* be stable if the steering axis tilt, mechanical trail, and forwards location of the front center of mass from the steering axis are all zero. We then show how stabilization is possible by any *one* of these design parameters. In addition, it is shown with specific examples: that the 'basic' bicycle model with no angular momentum due to the spinning of the wheels can be made stable; and that a bicycle with negative trail can be stable.

For comparison purposes and to gain further insight for further research, the conclusions of previous works were then reviewed and it was found that most analyses of bicycle and motorcycle stability have only explored small variations from standard designs popular from their day, and have provided few general conclusions or design guidelines. Because of this, a PC-based method for quickly determining stability at all speeds for any design configuration was created. The program is contained in Appendix B. Sample program runs are contained in Appendix C.

II. BACKGROUND AND TERMINOLOGY

Background

The study of bicycle stability is a subset of the study of the dynamics of two-wheeled vehicles, which include bicycles, motorcycles, and scooters. Such vehicles are sometimes called single-track vehicles in the technical literature. Historically, the published works involving two-wheel vehicles began with research of bicycle dynamics. In the the early 1900's when motorized vehicles became a practical means of transportation, research in two-wheel vehicles focused more on the motorcycle. This change in emphasis is probably due to the higher speeds involved in motorcycles and hence, the increased possibility of more serious accidents.[1] Still, as indicated in the reference list of this thesis, the phenomenon of how a bicycle stays upright

[1] This is not to say that bicycle accidents can not be serious. Other motives for the switch in emphasis of research may be economic, or related to the perception that the bicycle is only a toy for children, while the motorcycle is a serious vehicle for adults.

has remained a topic of general interest and of modern day research.

Dynamics of two-wheel vehicles can be studied with or without rider control. A study with rider control would allow, for example, the rider to impose torques to the handlebars or change the position of the upper body relative to the bicycle, typically in response to the bicycle's motion (feedback control). The dynamical equations developed in this thesis may be used to develop equations that include control, however, the analysis presented here neglects rider control/feedback.[2] In this thesis we are studying bicycle self-stability for a model with a rigid rider who is rigidly fixed to the rear frame of the bicycle and has no control over the bicycle.

As is indicated by the reference list to this thesis, recent technical works have focused on motorcycle dynamics and in particular have concentrated on developing tire models (see SAE Motorcycle Dynamics and Handling [1978]). However, as indicated by works of by Lowell and McKell [1983] and Daniel Kirshner [1982], the topic of bicycle dynamics is still not clearly understood.

Terminology

Throughout this thesis there are numerous terms that the reader who is not familiar with the mechanical layout of the bicycle may not understand. Figure 2.1 shows a typical bicycle with arrows labeling design parameters and components commonly used to describe bicycle geometry.

To clarify some terms used in this thesis we define them here:

1) trail — the distance, measured with the bicycle in an upright position, from

[2] See the Chapter I for rationale of why rider control/feedback is neglected.

the intersection of the steering axis with the ground back to the contact point of the front wheel.

2) mechanical trail — measured the same as trail, this is the perpendicular distance from the steering axis to the contact point of the front wheel.

3) rake (fork rake) — the perpendicular distance from the steering axis to the center of the front wheel.

4) tilt of steering axis (90° − headangle) — the angle defined by the steering axis and the vertical.

5) sliding (skidding) — when a point on the tire/wheel which is touching the ground moves relative to the ground.

6) sideslip — when a tire/wheel 'contact point' moves sideways relative to the ground from the direction the wheel is headed due to the elasticity in the tires and/or sliding. In discussion of vehicle dynamics the phrase 'sideslip' usually refers to the sideways motion due primarily to elasticity of the tires. When there is sideslip the direction of the velocity of the contact point is, in general, not in the plane of the wheel.

7) slip angle — the angle made with the tangent to the contact point path (instantaneous velocity direction) and the line from the intersection of the plane of the front wheel and the ground plane.

8) lean (tip, roll), steer, yaw, lateral displacement, and forward displacement — these are words for generalized coordinates and auxiliary variables used to describe the motion of the bicycle.

Some technical words used in discussion are:

9) generalized coordinates — minimal set of variables used to describe the position and orientation of the bicycle. The term generalized coordinates is usually only referred to when using Lagrange's equations.

10) holonomic constraint - a constraint which is integrable. That is, it is such that it is equivalent to some generalized coordinate being held constant, so the number of degrees of freedom reduces. Many treatise on analytical dynamics only treat constraints that are holonomic. It can be added to the derivation of the equations of motion at any time.

11) nonholonomic constraint - a constraint that is not integrable. A nonholonomic constraint does not restrict the configurations which may be achieved, but does reduce the number of ways the system is free to move at any instant. (Most typical is a wheel or a skate.) Generally, this type of constraint can be added to the derivation of the equations of motion only after Lagrange's equations (or equivalent) have been derived.

12) auxiliary variables - additional, redundant, variables used to describe, more conveniently, the position and orientation of the front part of the bicycle.

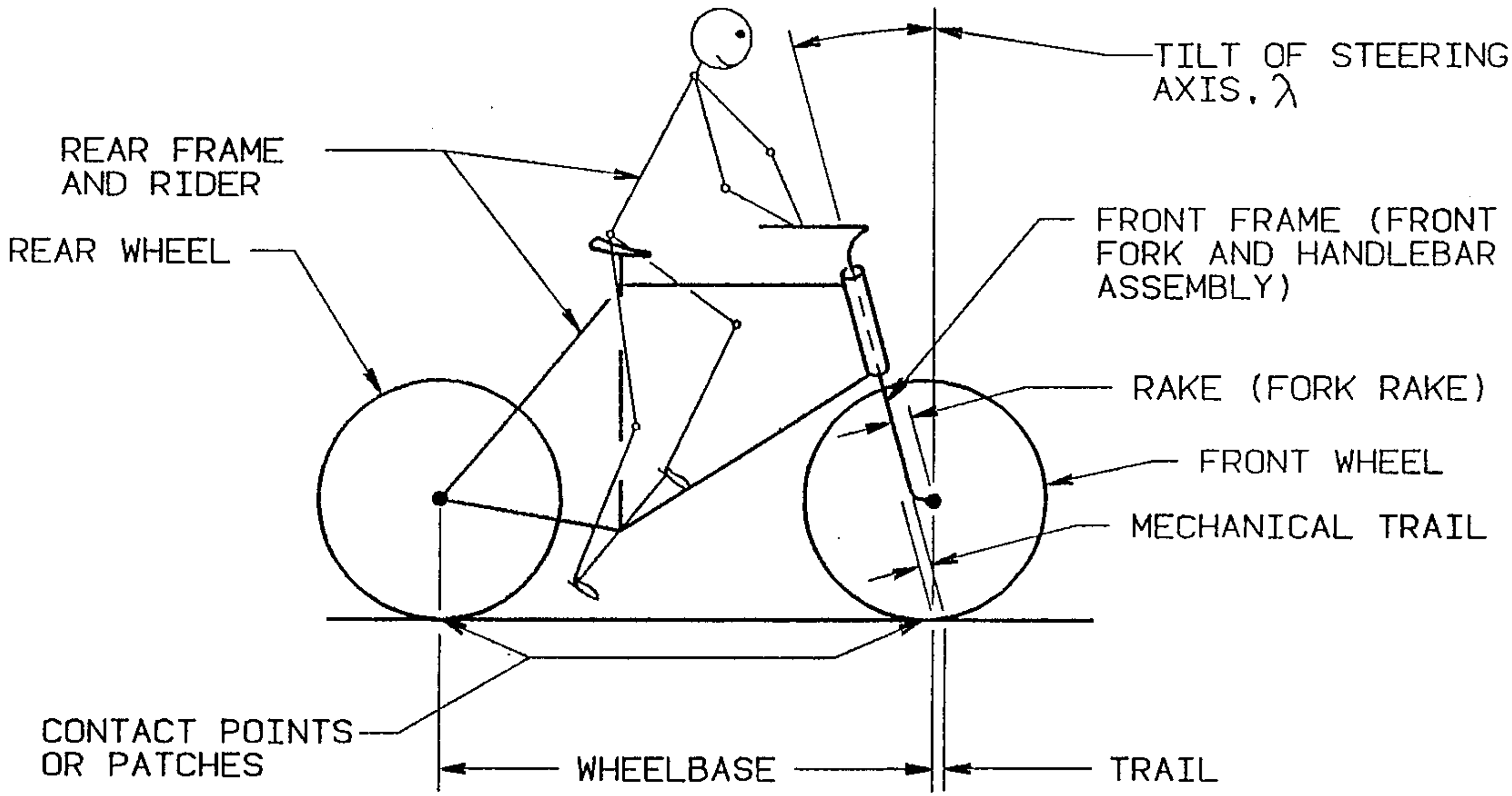
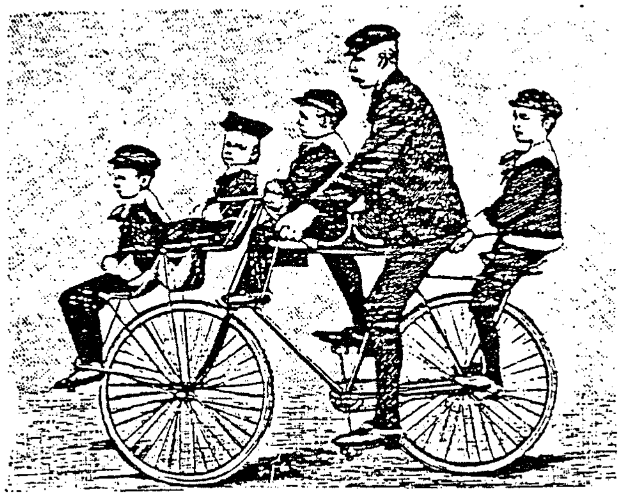


FIGURE 2.1

Some design parameters and components commonly used to describe bicycle geometry.



III. OVERVIEW OF THE DERIVATION OF THE LINEARIZED EQUATIONS OF MOTION FOR THE BASIC BICYCLE

Introduction

This chapter reviews the derivation of the linearized equations of motion for a bicycle model. In developing the model, simplifications were made to a real life bicycle by assumptions about the rider-bicycle system behavior. The rider-bicycle model used in this derivation was chosen to make passive rider (rigid rider, riding no-hands) stability analysis as simple as possible, without neglecting what were felt as the major stability related design parameters that affect the bicycle's 'self-stability'. The model has been used by several others, supporting our view point, and thus makes comparison to their equations of motion possible. Throughout this thesis this bicycle model is referred to as the *Basic bicycle model.* In Chapter VI of this thesis we will suggest ways to modify the Basic bicycle model by adding non-standard features such as gyroscopes, dampers, and springs to enhance or reduce the self-stability of the bicycle. We refer to these models as *Augmented bicycle models*.

The dynamical equations derived for the Basic bicycle model are not limited to the passive rider case, though that is the only case we investigate. As will be noted at the end of this chapter, these equations can be easily modified to incorporate rider-controlled steering torques.

The overview of equations and derivation technique for a Basic bicycle model was required because no commonly accepted set of equations have been established in the literature on bicycle stability. Many papers exist which present equations of motion.[1] However, few if any, close comparisons have been made to see if past derivation results agree. This chapter tries to clarify the notation and procedures for deriving the set of linearized equations of motion for the Basic bicycle model. Chapter IV then compares these results to past works.

In the derivation of the equations of motion we assume small deviations away from the vertical equilibrium position to make linearization of the equations possible. Our method of derivation uses Lagrange's equations with nonholonomic rolling constraints to arrive at the Basic bicycle model's linearized governing equations. The discussion follows the major steps taken in the derivation given by Neĭmark and Fufaev [1967]. However, we correct the potential energy expression presented by Neĭmark and Fufaev and point out other mistakes made in their derivation. In addition, we have tried to give further insight into the equations and their physical

[1] See Chapter IV.

meaning, and have simplified the algebra in presenting the final Basic bicycle model equations of motion. As far as we know, this is the first set of equations derived using Lagrangians equations which agrees in full with other Newtonian derivations, which adds to the presented equations credibility.

For a linearized Newtonian derivation the reader is referred to Döhring [1955] or Weir [1972], whose equations are equivalent linear combinations of what is presented here.[2] Döhring gives a good physical description of the equations of motion in his paper. Weir's equations are written for easy adaptation to control studies.

The derivation itself has several features that make it difficult to perform. First, the Basic bicycle is composed of four interconnected rigid bodies. This complicates the Newtonian analysis significantly, and therefore the Lagrangian formulation was used. Second, the constraints in this problem are nonholonomic; a topic not commonly studied or discussed even in advanced dynamics courses. Third, some expressions for kinetic and potential energy can not be solved for in closed form, so small angle approximations must be made before the full equations are derived, i.e. straight forward linearization from nonlinear equations is not possible. As a result, many previously derived sets of equations seem to be in error and we have tried to clarify points of past confusion.

Because we feel the method used in deriving the equations of motion is a major source of errors and confusion in the literature, throughout this chapter specific details of the derivation, its theoretical justification, and physical interpretation are

[2] See Chapter IV for exact conversion.

kept to a minimum in order to present the derivation in a compact form. Additional understanding of the equations and specific details of the steps involved are contained in Appendix A which is referred to throughout this chapter.

A number of potentially important issues regarding the linearization and the simplifications of the bicycle are actually quite subtle and perhaps important (such as tire radius and tire-road interaction).[3] Many are not studied in depth in this thesis, but some are discussed briefly in the text when appropriate. It is hoped that by clarifying the derivation for the Basic bicycle model, at least from a Lagrangian standpoint, these questions can be more readily approached.

This chapter is divided into the major steps that are taken in deriving the equations of motion for the Basic bicycle model using Lagrange's equations. These are: model description, definition of generalized coordinates and auxiliary variables, simplifying the nonholonomic rolling constraints, approximating the kinetic and potential energy to quadratic order, and finally applying the Lagrange operator to develop Lagrange's equations. With the aid of the simplified rolling constraints, the Basic bicycle model's equations of motion are developed by eliminating generalized forces of constraint from Lagrange's equations. In all steps the order of the equations used, for a system slightly perturbed from vertical straight-ahead equilibrium motion, is such that the final equations are correct to first order.

[3] See Kane [1977].

The Basic Bicycle Model

Assumptions

In building the Basic bicycle model we make the following simplifying assumptions about the bicycle-rider system and its behavior:

1. The bicycle consists of four rigid bodies as shown in figure 3.1: the rear frame with fixed rigid rider; the front fork/handlebar assembly; and the front and rear wheel. The wheels are considered to be rigid knife-edge disks.
2. The bicycle wheels roll on a rigid flat horizontal surface with enough friction between the wheels and the road to prevent sliding.
3. The rider does not move relative to the frame. (The rider may apply steering torques.)
4. Only small disturbances from the vertical straight-ahead equilibrium motion position are considered.
5. Within the bicycle-rider system there is no friction, e.g. no bearing torques between the wheels and wheel axle.
6. The only external forces applied to the bicycle are: a) body forces due to gravity b) the constraint forces from the ground on the wheels (keeping the bicycle from penetrating the ground and from slipping). c) a tipping torque acting on the rear frame. (This torque is set to zero for the passive-rider analysis. We included to it here to emphasize the equations' symmetry.)

7. An internal steering torque on the handlebar, reacted by the frame, is permitted. This torque is set to zero for passive-rider analysis.
8. When in its vertical equilibrium position the bicycle-rider system is symmetric about the vertical plane passing lengthwise through the rear frame.
9. We could assume constant velocity, though constant velocity is a consequence of the linearized equations.

The first two assumptions constitute a tire model having no sideslip angle. With the first assumption we have neglected any deformation of the rear frame, wheels, and front fork/handlebar assembly, and any motion of the rider relative to the bicycle. From the practical standpoint, we are also neglecting the motion of the chain and crank assembly, along with the pedaling motion of the riders legs. Most importantly, we are including the tires as part of the rigid *knife-edge* disk wheels (no tire radius), and because by definition they do not deform, no sideslip due to tire deformation is permitted and there is only one point on a wheel in contact with the ground.[4] The first assumption also implies that there is no energy loss due to tire hysteresis and that the contact point is always in the plane of the wheels.

The second assumption is critical in the derivation of the rolling constraints. It says that the front and rear contact point do not skid over the ground (no sliding). It is because of assumptions one and two that we can express the velocity of the location of the rear and front contact point in an inertia reference frame as functions of the orientation and spin rate of the rear and front wheel, respectively.

[4] For the definitions of slip angle, sideslip and sliding see Chapter II.

With the third assumption, we are neglecting any motion of the rider has relative to the bicycle (desired or undesired). This includes pedaling motion, arm motion to move the handlebars to steer the bicycle and any wiggling motion the rider may use to stay up with no hands.

The fourth assumption allows us to linearize the equations of motion and thus simplifies the form of the final equations of motion.

The fifth assumption also simplifies the equations of motion and makes applying Lagrange's equations easier.

Assumption six means we are neglecting any wind and air resistance the bicycle-rider system has. The validity of this assumption is decreased for motorcycle design due to higher speeds and greater air resistance; such forces may well be important for stability depending on how they vary with lean and steer angle, and how much weight transfer they produce.

Assumption seven has been added so that the final equations of motions are in a form that allows rider control of the steering to be added. Later, the internal torque is to zero for our passive rider analysis.

The eighth assumption significantly simplifies the kinetic energy expressions for the Basic bicycle model. It says that there is an equal distribution of mass on both sides of a plane that splits the bicycle model vertically while in its vertical equilibrium.

Alternative Way to View the Basic Bicycle Model

Equivalently, we can think of the development of the Basic model in a four step process. Figure 3.2a-d are a sequence of four pictures developing graphically the Basic bicycle model. The pictures are conceptual drawings of the Basic bicycle model based on the simplifying assumptions in our model. One interesting fact worth noting, is that the quadratic order geometrical relations for the Basic bicycle model are not dependent on the wheel radii, so the wheel contacts may be represented as specific points on the front and rear frames for the linearized equations. Also, for constant speed bicycle motion, the wheel angular momentum may be replaced with a constant speed gyrostat. (A gyrostat is a quantity of spin angular momentum not associated with any re-orientation inertia, so it reacts with a torque proportional to precession. It may be approximated physically by a very compact gyroscope spinning very fast.)

Reference Frames, Generalized Coordinates, and Auxiliary Variables

Reference Frames and Coordinate Axes

In order to derive the equations of motions of the bicycle most simply we introduce five reference frames. These reference frames are used to measure the position, velocity, and acceleration of the rear and front contact points, and measure the inertia of the rigid bodies that make up the Basic bicycle model. To clarify the notation we use the subscript r to refer to the rear part (rear frame + rider + rear wheel), the subscript f to refer the front part (front fork/handlebars assembly + front wheel), and let $\overline{(\ )}$ mean 'center of mass'. The coordinate systems used with each reference frame are shown in figures 3.3 and 3.4 and described as follows:

1. The inertial reference frame XYZ fixed to the ground with origin at point O.
2. A moving reference frame $x_r y_r z_r$ fixed to the rear frame with origin at P_r, the rear contact point.
3. A moving reference frame $x_f y_f z_f$ fixed to the front fork/handlebars assembly with origin at P_f, the front contact point.
4. A moving reference frame $\overline{x}_r \overline{y}_r \overline{z}_r$ fixed to the rear frame with origin at $\overline{P}_r$, the center of mass of the rear part of the bicycle (rear frame + rider + rear wheel).
5. A moving reference frame $\overline{x}_f \overline{y}_f \overline{z}_f$ fixed to the front fork/handlebar assembly with origin at $\overline{P}_f$, the center of mass of the front part of the bicycle (front fork/handlebar assembly + front wheel).

With the bicycle in the upright straight ahead configuration, the y_r and y_f axes are parallel to the XY plane, and the x_r and x_f axes are perpendicular to the rear frame, and, z_r and z_f are perpendicular to the ground, straight up. The orientation of these axes relative to the bicycle are shown in figures 3.3 and 3.4.

Generalized Coordinates and Auxiliary Variables

We are studying small deviations of a bicycle from the vertical equilibrium with approximately straight ahead motion initially along the Y axis, on a level plane, with approximately constant velocity.[5] The Basic bicycle model has seven independent generalized coordinates which describe its position and configuration for this case. We will next define these seven generalized coordinates, and then

[5] Later, we prove that for small lateral perturbations the velocity remains constant constant to first order.

introduce four auxiliary variables that will be used to simplify the derivation of the equations of motion. In section 1 of Appendix A we derive the relations between the generalized coordinates and auxiliary variables and show the how relationships are dependent on the geometry of the bicycle. The major geometric design parameters are shown in figure 3.5 and are defined throughout the chapter as needed.

The seven independent generalized coordinates which describe the bicycle position are as follows: the coordinates X_r and Y_r, measured from the inertial coordinate system, which locate the contact point P_r of the rear wheel with the ground; the angle θ_r, the heading of the rear frame with respect to the OY axis (the angle between the intersection of the rear frame plane with the XY plane, and the OY axis); the angle χ_r, the tilt of the rear frame from the vertical; the angle ψ, measured from the straight ahead position, through which the front fork/handlebar assembly fork is turned; and the angles ϕ_r and ϕ_f which describe the degree of rotation of the rear and front wheel respectively, measured relative to lines from the wheel centers to the contact points. When these generalized coordinates are specified, the position and orientation of the bicycle is known exactly with respect to the inertial coordinate system. The seven independent generalized coordinates are shown in figure 3.6. With reference to a forward facing rider with the bicycle moving predominantly along the inertial OY axis (where Z is the vertical), the positive sign conventions are given by: χ_r, tilted over to the right; ψ, steer to the left; θ_r, headed to the left; ϕ_r and ϕ_f, angle due to forward roll.[6] All angles in figure 3.6 are shown in their

[6] It should be noted that the definition of ϕ_r and ϕ_f are slightly irregular, in that, positive $\dot{\phi}_r$ and positive $\dot{\phi}_f$ imply a wheel angular velocity vector in the negative

positive direction.

In the literature the variables X_r, χ_r, θ_r, and ψ are often referred to as lateral motion, roll or lean, yaw or heading, and steer. We will use both terminologies where convenient.

Also, for convenience and added clarity in the derivation, we introduce four *dependent* front coordinates X_f, Y_f, θ_f, and χ_f which locate the front contact point, P_f, and give the heading and lean of the front assembly, in the same manner as their respective rear coordinate counterparts. For the given assumptions in the Basic bicycle model they can be derived to quadratic order as functions of the generalized coordinates. Hence we will call them *auxiliary variables*. These variables have the same sign convention as those of the rear coordinates and are also shown in figure 3.6.

Relations Between the Generalized Coordinates and Auxiliary Variables

In section 1 of Appendix A, methods for deriving the relations between the generalized coordinates and auxiliary variables are shown. The results to first order for the case where the bicycle's frame maintains small deviations from the YZ plane

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x_r and x_f direction, respectively.

take the form,[7]

$$X_f - X_r = -c_w\theta_r + c_f\psi \tag{3.1a}$$

$$Y_f - Y_r = c_w \tag{3.1b}$$

$$\chi_f - \chi_r = -\psi\sin\lambda \tag{3.1c}$$

$$\theta_f - \theta_r = \psi\cos\lambda \tag{3.1d}$$

where c_w, λ, and c_f are defined as,

c_w — the wheelbase, distance between P_r and P_f.

λ — the steering axis tilt, measured from the vertical towards the rider ($90°-$ headangle).

c_f — perpendicular distance behind steering axis to P_f.

These quantities are geometric design parameters of the bicycle and shown in their positive configuration in figure 3.5. All are measured when the bicycle is in its vertical straight ahead position. Later in Chapter V it will be shown how these parameters affect the stability of the Basic bicycle model.

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[7] Because $\dot{Y}_r$ is not a small quantity it is technically necessary to express equation (3.1b) to second order when developing the kinetic energy as follows,

$$Y_f - Y_r = c_w\big(1 - \frac{\theta_r^2}{2}\big) + c_f\frac{\psi^2}{2}\cos\lambda$$

so that,

$$\dot{Y}_f = \dot{Y}_r - c_w\dot{\theta}_r\theta_r + c_f\cos\lambda\dot{\psi}\psi \tag{3.1b$'$}$$

This is used when calculating the kinetic energy of the front part of the bicycle where $\dot{Y}_f^2$ is required. However, including second order terms of the form $(\dot{q_i q_i}) = 2q_i\dot{q}_i$ in the Lagrangian of second order does not affect Lagrange's equations. Hence, although technically required, the additional terms do not contribute to Lagrange's equations.

The equations (3.1a-d) are used in simplifying the rolling constraint equations and kinetic energy of the front part of the Basic bicycle model. They are referred to as the relations between the generalized coordinates and the auxiliary variables throughout this thesis. For the derivation and a better understanding of these relations the reader is referred to section 1 of Appendix A.

Rolling constraints

In general a bicycle has four nonholonomic contraints relating the motion of the location of the rear and front contact point in the inertial XY reference plane, to the orientation and rotation rate of the wheels. These relations are referred to as rolling constraints. We can specify these rolling constraints due to our assumptions of no 'sliding' (skidding) and no 'sideslip' (due to tire deformation). These relations can be linearized according to our assumption of only small deviations from the equilibrium motion, because they are added to the problem only *after* the Lagrange equations are developed. The relevant first-order results, which are used later to develop the equations of motion, and which are derived in more detail in section 2 of Appendix A, can be expressed as follows,

$$\ddot{Y}_r = a_r \ddot{\phi}_r \tag{3.2a}$$

$$\ddot{Y}_r = a_f \ddot{\phi}_f \tag{3.2b}$$

$$\dot{\theta}_r = \frac{c_f}{c_w}\dot{\psi} + \psi \dot{Y}_r \frac{\cos\lambda}{c_w} \tag{3.2c}$$

$$\ddot{\theta}_r = \frac{c_f}{c_w}\ddot{\psi} + \dot{\psi}\dot{Y}_r \frac{\cos\lambda}{c_w} + \psi \ddot{Y}_r \frac{\cos\lambda}{c_w} \tag{3.2d}$$

$$\ddot{X}_r = -\ddot{Y}_r\theta_r - \dot{Y}_r\dot{\theta}_r = -\ddot{Y}_r\theta_r - \frac{c_f}{c_w}\dot{Y}_r\dot{\psi} - \psi\dot{Y}_r^2\frac{\cos\lambda}{c_w} \tag{3.2e}$$

Later equations (3.2a,b) are used to prove that for small disturbances, $\ddot{Y}_r = 0$ to first order, so it can be eliminated from (3.2d,e).

Originally the number of generalized coordinates is seven. By implementing the constraint equations (3.2a-e), four of which are independent, the variables θ_r, X_r, ϕ_r, and ϕ_f will be eliminated from Lagrange's equations, and thus the system will be left with only three generalized coordinates: χ_r, the lean angle, ψ, the steer angle, and Y_r, the coordinate that locates the rear wheel along the Y axis. These three generalized coordinates represent the three degrees of freedom for the linearized Basic bicycle model. However, as just mentioned $\ddot{Y}_r$ is zero to first order and therefore $\dot{Y}_r$ becomes a constant to first order.[8] Thus because Y_r is not present in the final equations there are only *two nontrivial* degrees of freedom for the linearized model. Mathematically this means that two second order differential equations, or one fourth order differential equation, represent the motion of the system.

In general, these constraints should be added to the problem only after Lagrange's equations have been found. Also, note that the generalized coordinate χ_r is not present in the rolling constraint relations and that the relations are not dependent on the wheel radii.

Kinetic Energy of the Basic Bicycle Model

Because Lagrange's equations involve taking the derivative of the Lagrange function, in order to formulate the equations of motion of the Basic bicycle model to first order, the total kinetic energy of the Basic bicycle model is needed to second

[8] This is one of the subtle points we referred to earlier.

order. However, as discussed in section 3 of Appendix A, simplifications can be made to the kinetic energy expression by eliminating terms that do not contribute when formulating Lagrange's equations. Terms that are constant or which are time derivatives of the product of two generalized coordinates, such as $\dot{(\theta_r\theta_r)} = 2\dot{\theta}_r\theta_r$ and $\dot{(\chi_r\theta_r)} = \dot{\chi}_r\theta_r + \theta_r\dot{\chi}_r$, do not contribute to when formulating Lagrange's equations. As a result, any term with the following coefficients can be left out due to the fact that when Lagrange's equations are developed it does not contribute: $\dot{(\psi\theta_r)}$, $\dot{(\chi_r\theta_r)}$, $\dot{\theta}_r\theta_r$, $\dot{\psi}\psi$. This chapter only presents the contributing terms of the kinetic energy. For discussion on how to include the additional noncontributing terms refer to section 3 of Appendix A.

To calculate the total kinetic energy, the bicycle can be broken into two separate pieces, the rear part (rear frame + rider + rear wheel) and the front part (front fork/handlebar assembly + front wheel). Here we denote the respective kinetic energies as KE_r^+ and KE_f^+, where the $(\ )^+$ means 'contributing terms only'. This section presents the result of KE_r^+ which is derived in more detail in section 3 of Appendix A. KE_f^+ is then derived from KE_r^+ using the relations between the generalized coordinates and auxiliary variables, equations (3.1a-d).

Kinetic Energy of the Rear Part of the Basic Bicycle Model

The result from section 3 of Appendix B of the contributing terms of KE_r to second order is:

$$KE_r^+ = \overbrace{\frac{1}{2}m_r(\dot{X}_r - \overline{l}_r\dot{\theta}_r + \overline{h}_r\dot{\chi}_r)^2 + \frac{1}{2}m_r\dot{Y}_r^2}^{KE_r^{trans}}$$

$$+\overbrace{\frac{1}{2}(R_{\overline{yy}}\dot{\chi}_r^2 + 2R_{\overline{yz}}\dot{\chi}_r\dot{\theta}_r + R_{\overline{zz}}\dot{\theta}_r^2)}^{KE_r^{rot}}$$

$$+\overbrace{C_r\dot{\phi}_r\chi_r\dot{\theta}_r + \frac{1}{2}C_r\dot{\phi}_r^2}^{KE_{rwheel}^{spin}} \tag{3.3}$$

where,

m_r	—	mass of the rear part of the Basic bicycle (frame+ rider + wheel).
$\overline{l}_r, \overline{h}_r$	—	length and height to the rear center of mass measured in $x_r y_r z_r$.
$R_{\overline{yy}}$, $R_{\overline{yz}}$, $R_{\overline{zz}}$	—	components of the rear part inertia tensor measured in $\overline{x}_r\overline{y}_r\overline{z}_r$ (includes the rear wheel).
C_r	—	polar mass moment of inertia of the rear wheel.

The inertia tensor terms $R_{\overline{yy}}$, $R_{\overline{yz}}$, and $R_{\overline{zz}}$ are the components of a matrix representing the rear inertia about the axes with origin at the rear center of mass $\overline{P}_r$, measured in the $\overline{x}_r\overline{y}_r\overline{z}_r$ as shown in figure 3.4. The rear inertia tensor is then,

$$\begin{pmatrix} R_{\overline{xx}} & 0 & 0 \\ 0 & R_{\overline{yy}} & R_{\overline{yz}} \\ 0 & R_{\overline{zy}} & R_{\overline{zz}} \end{pmatrix}_{\overline{x}_r\overline{y}_r\overline{z}_r}$$

where $R_{\overline{xy}} = R_{\overline{xz}} = 0$ because of our assumption of symmetry about the plane of the rear frame, and $R_{\overline{yz}} = R_{\overline{zy}}$ by definition of the inertia tensor components. To simplify the derivation, the rear inertia includes the inertia of the rear wheel as if it where glued to the rear frame.

It should be emphasized that the term $R_{\overline{yz}}$ is a component of the rear inertia *tensor* defined as

$$R_{\overline{yz}} = -\int \overline{y}\,\overline{z}\,dm$$

This should not be confused with the *product mass moment of inertia* term $I_{\overline{yz}}$, which is equal to the $\int \overline{y}\,\overline{z}\,dm$. In the derivations and notation used in this thesis, the off-diagonal inertia terms are always components of an inertia tensor, *not* the product mass moment of inertia.

The overbraces in equation (3.3) indicate where terms under the braces came from when deriving the contributing kinetic energy. KE_r^{trans} means these are terms from the rigid body translation kinetic energy of the rear. KE_r^{rot} means the terms come from rigid body rotation of the rear part.[9] And KE_{rwheel}^{spin} means these terms were derived from the kinetic energy due to the rotation of the rear wheel relative to the frame, the so-called 'spin' of the wheel.

Kinetic Energy of the Front Part of the Basic Bicycle

The kinetic energy of the front part of the bicycle can be derived from that of the rear. Hence, by replacing the r subscripts in equations (3.3) with a f for front the kinetic energy of the front part of the bicycle is obtained as,

$$\begin{aligned} KE_f^{+} &= \overbrace{\frac{1}{2}m_f(\dot{X}_f - \overline{l}_f\dot{\theta}_f + \overline{h}_f\dot{\chi}_f)^2 + \frac{1}{2}m_f\dot{Y}_f^2}^{KE_f^{trans}} \\ &\quad + \overbrace{\frac{1}{2}(F_{\overline{yy}}\dot{\chi}_f^2 + 2F_{\overline{yz}}\dot{\chi}_f\dot{\theta}_f + F_{\overline{zz}}\dot{\theta}_f^2)}^{KE_f^{rot}} \\ &\quad + \overbrace{C_f\dot{\phi}_f\chi_f\dot{\theta}_f + \frac{1}{2}C_f\dot{\phi}_f^2}^{KE_{fwheel}^{spin}} \end{aligned} \tag{3.4}$$

Just as for the contributing rear kinetic energy, equation (3.3), in equation

[9] As discussed in section 3 of Appendix A, we treat the rear wheel as if it is glued to the rear frame and then add terms due to the spinning of the wheels separately.

(3.4) some terms have been omitted because they do not contribute to Lagrange's equations. Equation (3.4) is not in terms of the generalized coordinates, which we need to derive Lagrange's equations. Using the relations between the generalized coordinates and the auxiliary variables, equations (3.1a-d), we can expand equation (3.4) in terms of the generalized coordinates. In doing so, we *again* omit terms that do not contribute to Lagrange's equations which yields the contributing kinetic energy of the front in terms of the generalized coordinates,[10]

$$
\begin{aligned}
KE_f^{++} = &\ \overbrace{\frac{1}{2}m_f\Big(\dot{X}_r - (c_w + \bar{l}_f)\dot{\theta}_r + \bar{h}_f\dot{\chi}_r - d\dot{\psi}\Big)^2 + \frac{1}{2}m_f\dot{Y}_r^2}^{KE_f^{trans}} \\
&+ \overbrace{\frac{1}{2}\Big(F_{\overline{yy}}(\dot{\chi}_r - \dot{\psi}\sin\lambda)^2 + 2F_{\overline{yz}}(\dot{\chi}_r - \dot{\psi}\sin\lambda)(\dot{\theta}_r + \dot{\psi}\cos\lambda) + F_{\overline{zz}}(\dot{\theta}_r + \dot{\psi}\cos\lambda)^2\Big)}^{KE_f^{rot}} \\
&+ \overbrace{C_f\dot{\phi}_f\Big(\chi_r\dot{\theta}_r - \dot{\theta}_r\psi\sin\lambda + \chi_r\dot{\psi}\cos\lambda\Big) + \frac{1}{2}C_f\dot{\phi}_f^2}^{KE_{f\,wheel}^{spin}} \qquad (3.5)
\end{aligned}
$$

where,

$$d = \bar{h}_f\sin\lambda + \bar{l}_f\cos\lambda - c_f$$

Because of the need to omit noncontributing terms from equation (3.4) in terms of the generalized coordinates equations, $KE_f^{+} \neq KE_f^{++}$. For this reason we have made a double plus superscript for the contributing kinetic energy of the front part in equation (3.5).

It can easily be shown that d is the perpendicular distance from the center of mass of the front part to the steering axis as shown in figure 3.7. Once again, as in

[10] Note that in equation (3.4), $\dot{Y}_f$ is not constant to second order. However, second order terms of this function are noncontributing as is shown in (3.1b)$'$.

equation (3.3) for the rear part, the overbraces in equation (3.5) indicate where the terms originate in the derivation of the kinetic energy.

The total contributing kinetic energy KE_t^+ of the basic bicycle model is then the addition of equations (3.3) and (3.5),

$$KE_t^+ = KE_r^+ + KE_f^{++} \tag{3.6}$$

We emphasize that only the terms that contribute to Lagrange's equations have been kept in these expressions.

Especially note the total contributing kinetic energy has been expressed in terms of the seven generalized coordinates Y_r, ϕ_r, ϕ_f, X_r, χ_r, θ_r, and ψ. All auxiliary variables have been eliminated. Later, *after* developing Lagrange's equations, with the help of the rolling constraints (3.2a-b), we will prove that for small disturbances, $\dot{Y}_r$, $\dot{\phi}_r$, and $\dot{\phi}_f$, are all constant to first order, thus simplifying the equations of motion further. With the help of the rolling constraints (3.2c-e), we will reduce the number of generalized coordinates in the equations to two, by further eliminating the variables χ_r and θ_r from the equations of motion.

Potential Energy of the Basic Bicycle[11]

Using the inertial reference system XYZ the total potential energy of the Basic bicycle, PE_t, at any instant is by definition,

$$PE_t = m_r g \overline{Z}_r + m_f g \overline{Z}_f \tag{3.7}$$

[11] The results of this section differ from those of Neĭmark and Fufaev [1967].

where $m_r g\overline{Z}_r$ is the potential energy of the rear, $m_f g\overline{Z}_f$ the potential energy of the front, and g is the gravitational constant. The heights $\overline{Z}_r$ and $\overline{Z}_f$ of the rear and front center of mass respectively are then measured from the XY plane, i.e. the road surface as shown in figure 3.8. In general, $\overline{Z}_r$ and $\overline{Z}_f$ change only due to variations of the lean angle χ_r, and the steering angle ψ, and are independent of the generalized coordinates, X_r, Y_r, θ_r, ϕ_r and ϕ_f. Thus the potential energy is only a function of the lean and steer angle, χ_r and ψ, respectively. Section 4 of Appendix A presents the derivation of the potential energy to second order, i.e. the details of how $\overline{Z}_r$ and $\overline{Z}_f$ change as a function of χ_r and ψ. The results are that the total potential energy of the bicycle can be written to second order as,

$$\begin{aligned} PE_t =& \overbrace{m_r g\overline{h}_r + m_f g\overline{h}_f}^{constant} - \frac{g}{2} m_t \overline{h}_t \chi_r^2 \\ & - \frac{g m_f d}{2}\left(\psi^2 \sin\lambda - 2\chi_r\psi\right) \\ & - \frac{g\overline{l}_t m_t c_f}{2c_w}\left(\psi^2 \sin\lambda - 2\chi_r\psi\right) \end{aligned} \qquad (A.20a)$$

where,

$$m_t = m_r + m_f$$

$$\overline{l}_t = \frac{m_r\overline{l}_r + m_f(c_w + \overline{l}_f)}{m_t}$$

This can be simplified by defining ν to be the following,

$$\nu = m_f d + \frac{c_f m_t \overline{l}_t}{c_w}$$

Then the contributing potential energy (leaving out the constant terms $m_r g\overline{h}_r$ and $m_f g\overline{h}_f$) becomes,

$$PE_t^+ = -\frac{g}{2}(m_t\overline{h}_t\chi_r^2 - 2\psi\chi_r\nu + \psi^2\nu\sin\lambda) \qquad (3.8)$$

where again $(\ )_t$ stands for total, $(\ )^+$ again means 'contributing terms', and $\bar{l}_t$ which is the distance forward from the rear contact point, P_r, to the center of mass for the total bicycle, measured in the plane of the rear frame as shown in figure 3.9.

Lagrange's Equations for the Basic Bicycle Model

We now introduce the Lagrange function L,

$$L = KE_t^+ - PE_t^+$$

where KE_t^+ and PE_t^+ are the kinetic and potential energy which contribute to Lagrange's equations and are given by equations (3.6) and (3.8). In order to derive Lagrange's equations we also introduce the Lagrangian operator $\mathcal{L}_q$,

$$\mathcal{L}_q = \frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}$$

where q represents the generalized coordinates Y_r, ϕ_r, ϕ_f, X_r, θ_r, χ_r, and ψ.

What is the quantity defined by $\mathcal{L}_q$ operating on L? As is explained by Goldstein [1980], the $\mathcal{L}_q$ operating on L is a generalized force on the system. In this case with the help of figure 3.10, the generalized forces represented by the letter Q, can be written to first order as follows,[12]

$$\mathcal{L}_{Y_r}\ (L) = Q_{Y_r} = -D_f - D_r \tag{3.9a}$$

$$\mathcal{L}_{\phi_r}\ (L) = Q_{\phi_r} = D_r a_r \tag{3.9b}$$

[12] In Figure 3.10 M_ψ is an internal torque to the Basic bicycle model imposed by the rider on the steering axis and reacted by the rear frame. In our free body diagram we have assumed the component of the reaction moment on the rear frame, say M_{θ_r}, is negligible.

$$\mathcal{L}_{\phi_f}\left(L\right) = Q_{\phi_f} = D_f a_f \tag{3.9c}$$

$$\mathcal{L}_{X_r}\left(L\right) = Q_{X_f} = F_f + F_r \tag{3.9d}$$

$$\mathcal{L}_{\theta_r}\left(L\right) = Q_{\theta_r} = -F_f c_w \tag{3.9e}$$

$$\mathcal{L}_{\chi_r}\left(L\right) = Q_{\chi_r} = M_{\chi_r} \tag{3.9f}$$

$$\mathcal{L}_{\psi}\left(L\right) = Q_{\psi} = M_{\psi} + c_f F_f \tag{3.9g}$$

where,

Q_q - the generalized forces conjugate to the various generalized coordinates (q represents the respective length or angle, and Q_q is thus a force or a moment).

D_r, D_f - the constraint forces on the rear and front wheel, respectively, initially in the Y direction.

F_r, F_f - the constraint forces on the rear and front wheel, respectively, initially in the X direction.

M_{χ_r}, M_{ψ} - the moments M_{ψ} and M_{χ_r} representing the steer torque and tipping torque, respectively.

G_r, G_f - the constraint forces on the rear and front wheel, respectively, always in the Z direction. To first order these forces always add to be equal and opposite to the forces due to gravity.

Rewriting $\mathcal{L}_q\left(L\right)$ in terms of the bicycle design parameters we have,

$$Q_{Y_r} = -D_f - D_r = m_t \ddot{Y}_r \tag{3.10a}$$

$$Q_{\phi_r} = D_r a_r = C_r \ddot{\phi}_r \tag{3.10b}$$

$$Q_{\phi_f} = D_f a_f = C_f \ddot{\phi}_f \tag{3.10c}$$

$$Q_{X_r} = F_f + F_r = m_t \ddot{X}_r - m_t \bar{l}_t \ddot{\theta}_r + m_t \bar{h}_t \ddot{\chi}_r - m_f d \ddot{\psi} \tag{3.10d}$$

$$\begin{aligned} Q_{\theta_r} =& - c_w F_f = -m_t \bar{l}_t \ddot{X}_r + T_{zz} \ddot{\theta}_r + T_{yz} \ddot{\chi}_r + H_t \dot{\chi}_r + F''_{\lambda z} \ddot{\psi} \\ & - H_f \sin\lambda \dot{\psi} \end{aligned} \tag{3.10e}$$

$$\begin{aligned} Q_{\chi_r} =& M_{\chi_r} = m_t \bar{h}_t \ddot{X}_r + T_{yz} \ddot{\theta}_r - H_t \dot{\theta}_r + T_{yy} \ddot{\chi}_r - g m_t \bar{h}_t \chi_r + F'_{\lambda y} \ddot{\psi} \\ & - H_f \cos\lambda \dot{\psi} + g\nu\psi \end{aligned} \tag{3.10f}$$

$$\begin{aligned} Q_{\psi} =& M_{\psi} + c_f F_f = -m_f d \ddot{X}_r + F''_{\lambda z} \ddot{\theta}_r + H_f \sin\lambda \dot{\theta}_r + F'_{\lambda y} \ddot{\chi}_r \\ & + H_f \cos\lambda \dot{\chi}_r + g\nu\chi_r + F'_{\lambda\lambda} \ddot{\psi} - g\nu \sin\lambda \psi \end{aligned} \tag{3.10g}$$

where,

$$H_r = C_r \dot{\phi}_r \tag{3.11a}$$

$$H_f = C_f \dot{\phi}_f \tag{3.11b}$$

$$H_t = H_f + H_r \tag{3.11c}$$

$$T_{yy} = m_r \bar{h}_r^2 + R_{\overline{yy}} + m_f \bar{h}_f^2 + F_{\overline{yy}} \tag{3.11d}$$

$$T_{zz} = m_r \bar{l}_r^2 + R_{\overline{zz}} + m_f (c_w + \bar{l}_f)^2 + F_{\overline{zz}} \tag{3.11e}$$

$$T_{yz} = -m_r \bar{h}_r \bar{l}_r + R_{\overline{yz}} - m_f \bar{h}_f (c_w + \bar{l}_f) + F_{\overline{yz}} \tag{3.11f}$$

$$F'_{\lambda\lambda} = m_f d^2 + F_{\overline{yy}} \sin^2\lambda - F_{\overline{yz}} \sin 2\lambda + F_{\overline{zz}} \cos^2\lambda \tag{3.11g}$$

$$F'_{\lambda y} = -m_f \bar{h}_f d - F_{\overline{yy}} \sin\lambda + F_{\overline{yz}} \cos\lambda \tag{3.11h}$$

$$F''_{\lambda z} = m_f (c_w + \bar{l}_f) d - F_{\overline{yz}} \sin\lambda + F_{\overline{zz}} \cos\lambda \tag{3.11i}$$

The quantities in equation (3.11a-i) seem to have clear physical significance,

H_r, H_f — The angular momentum due to spinning of the rear wheel and front wheel, respectively.

H_t — The total angular momentum of the bicycle due to wheel spinning.

T_{yy}, T_{yz}, T_{zz} — The inertia tensor components for the entire bicycle in its upright position, about the rear contact point in measured in $x_r y_r z_r$.

$F'_{\lambda\lambda}$ — The mass moment of inertia about the steering axis of the front part.

$F'_{\lambda y}$ — The quantity defined by the front inertia tensor measured about the point where the steering axis intersects the ground, dotted with unit vectors in the Y direction and λ direction. That is, it is the torque about the steering axis needed for lean-acceleration of the bicycle.

$F''_{\lambda z}$ — The quantity defined by the front inertia tensor measured about the intersection of the perpendicular from the rear contact with the steering axis, dotted with unit vectors in the Z direction and λ direction. That is, the torque about the steering axis due to yaw acceleration of the bicycle.

Further we define $\mathbf{T}$ the inertia tensor for the entire bicycle (with wheels glued to the frame) about the rear contact point, P_r. The tensor $\mathbf{T}$ is defined just like $\mathbf{R}$ and looks like

$$\begin{pmatrix} T_{xx} & 0 & 0 \\ 0 & T_{yy} & T_{yz} \\ 0 & T_{zy} & T_{zz} \end{pmatrix}_{x_r y_r z_r}$$

Linearized Equations of Motion for the Basic Bicycle Model

Equations of Motion

Using equations (3.9e,g,f), we can solve for two equations of motions for the Basic bicycle model in terms of Lagrange's equations, and, in addition, two equations which determine the lateral forces in the X_r direction on the front and rear wheel. Eliminating F_f from (3.9e) using (3.9g) and leaving (3.9f) alone we get the Basic bicycle model's equations of motion in terms of Lagrange's equations,

$$\mathcal{L}_{\chi_r}(L) = M_{\chi_r} \tag{3.12a}$$

$$\mathcal{L}_{\psi}(L) + \frac{c_f}{c_w}\mathcal{L}_{\theta_r}(L) = M_{\psi} \tag{3.12b}$$

When the generalized coordinates θ_r and X_r have been eliminated from Lagrange's equations using the constraint equations (3.2c-e), equations (3.12a,b) will be our two equations of motion governing leaning χ_r and steering ψ.

By eliminating F_f from equation (3.9d) and rewriting (3.9e) we have two equations that can be used to solve for the lateral forces on the Basic bicycle model,

$$\mathcal{L}_{X_r}(L) + \frac{\mathcal{L}_{\theta_r}(L)}{c_w} = F_r \tag{3.13a}$$

$$-\frac{\mathcal{L}_{\theta_r}(L)}{c_w} = F_f \tag{3.13b}$$

Once χ_r and ψ are found, equations (3.13a,b) may be used to find the lateral forces from the ground on the front and rear wheel, F_f and F_r, respectively.

Proof that Velocity Terms are Constant to First Order

Using equations (3.10b) and (3.10c) to eliminate D_r and D_f from (3.10a) we find,

$$\overbrace{\left(m_t + \frac{C_r}{a_r^2} + \frac{C_f}{a_f^2}\right)}^{\text{always positive}}\ddot{Y}_r = 0$$

Since, as indicated by the overbrace, the coefficient to $\ddot{Y}_r$ is always positive, $\ddot{Y}_r$ must be zero to first order and,

$$\therefore\ \ddot{Y}_r = 0$$

If $\ddot{Y}_r = 0$, then by using the first two rolling constraint equations presented in this chapter (3.2a-b) we can also prove that $\ddot{\phi}_r$ and $\ddot{\phi}_f$ are zero to first order, or $\dot{\phi}_r$ and $\dot{\phi}_f$ are constant. Thus, for small perturbations from the upright position, the linearized equations $\mathcal{L}_{Y_r}\ (L) = 0$, $\mathcal{L}_{\phi_r}\ (L) = 0$, and $\mathcal{L}_{\phi_f}\ (L) = 0$. That is, Y_r, ϕ_r, and ϕ_f are ignorable coordinates and can be eliminated from the equations of motion.[13]

Elimination of χ_r and θ_r from equations (3.12a,b)

Substituting the remaining rolling constraints (3.2c-e) (where $\dot{Y}_r$ has been set equal to a constant V and $\ddot{Y}_r = 0$),

$$\dot{\theta}_r = \frac{c_f}{c_w}\dot{\psi} + \psi V \frac{\cos\lambda}{c_w} \tag{3.2c}$$

$$\ddot{\theta}_r = \frac{c_f}{c_w}\ddot{\psi} + \dot{\psi} V \frac{\cos\lambda}{c_w} \tag{3.2d}$$

$$\ddot{X}_r = -V\dot{\theta}_r = -\frac{c_f}{c_w}V\dot{\psi} - \psi V^2 \frac{\cos\lambda}{c_w} \tag{3.2e}$$

into equations (3.12a,b), the variables $\dot{\theta}_r$, $\ddot{\theta}_r$, and $\ddot{X}_r$ can be eliminated. Thus we form two coupled second order linear differential equations with constant coefficients. These equations take the general form,

$$\mathcal{M}_{\chi\chi}\ \ddot{\chi}_r + \mathcal{K}_{\chi\chi}\ \chi_r + \mathcal{M}_{\chi\psi}\ \ddot{\psi} + \mathcal{C}_{\chi\psi}\ \dot{\psi} + \mathcal{K}_{\chi\psi}\ \psi = M_{\chi_r} \tag{3.14}$$

[13] The topic of ignorable coordinates and their physical meaning in a nonholonomic system is not clearly understood by the author. For more discussion on ignorable coordinates for holonomic systems see Wells [1967] pp. 235.

$$\mathcal{M}_{\psi\psi}\,\ddot{\psi}+\mathcal{C}_{\psi\psi}\,\dot{\psi}+\mathcal{K}_{\psi\psi}\,\psi+\mathcal{M}_{\psi\chi}\,\ddot{\chi}_r+\mathcal{C}_{\psi\chi}\,\dot{\chi}_r+\mathcal{K}_{\psi\chi}\,\chi_r=M_\psi \tag{3.15}$$

Equation (3.14) is the first order approximation of the moment, $\mathcal{M}_{\chi_r}$, about the track of the bicycle, applied to the rear frame, e.g. by training wheels. For this reason we call it the *lean* equation. The second equation (3.15) provides the torque about the steering axis, $\mathcal{M}_\psi$, thus we call it the *steer* equation.

Expanding equations (3.14) and (3.15) in terms of the bicycle parameters the coefficients to the lean equation are:

$$\mathcal{M}_{\chi\chi}=T_{yy}$$

$$\mathcal{C}_{\chi\chi}=0$$

$$\mathcal{K}_{\chi\chi}=-gm_t\bar{h}_t$$

$$\mathcal{M}_{\chi\psi}=F'_{\lambda y}+\frac{c_f}{c_w}T_{yz}$$

$$\mathcal{C}_{\chi\psi}=-\left(H_f\cos\lambda+\frac{c_f}{c_w}H_t\right)+V\left(T_{yz}\frac{\cos\lambda}{c_w}-\frac{c_f}{c_w}m_t\bar{h}_t\right)$$

$$\mathcal{K}_{\chi\psi}=g\nu-H_tV\frac{\cos\lambda}{c_w}-V^2\frac{\cos\lambda}{c_w}m_t\bar{h}_t$$

and the coefficients to the steer equation are:

$$\mathcal{M}_{\psi\psi}=F'_{\lambda\lambda}+2\frac{c_f}{c_w}F''_{\lambda z}+\frac{c_f^2}{c_w^2}T_{zz}$$

$$C_{\psi\psi}=V\left(\frac{\cos\lambda}{c_w}\left(F''_{\lambda z}+\frac{c_f}{c_w}T_{zz}\right)+\frac{c_f}{c_w}\nu\right)$$

$$\mathcal{K}_{\psi\psi}=-g\nu\sin\lambda+VH_f\sin\lambda\frac{\cos\lambda}{c_w}+V^2\nu\frac{\cos\lambda}{c_w}$$

$$\mathcal{M}_{\psi\chi}=F'_{\lambda y}+\frac{c_f}{c_w}T_{yz}$$

$$\mathcal{C}_{\psi\chi}=H_f\cos\lambda+\frac{c_f}{c_w}H_t$$

$$\mathcal{K}_{\psi\chi}=g\nu$$

Note that the coefficients are functions of velocity, dependent on wheel radii only in the relation of H_f and H_r to V. (See equations (3.11a, b) and note $V = \dot{\phi}_r a_r$.)

Representation of the Equations of Motion

Equations (3.14) and (3.15) can be expressed either as two second order differential equations or one fourth order differential equation. For the case of two second order differential equation it is common to write the equations in matrix form as follows,

$$\left[\overbrace{\begin{pmatrix} \mathcal{M}_{\chi\chi} & \mathcal{M}_{\chi\psi} \\ \mathcal{M}_{\psi\chi} & \mathcal{M}_{\psi\psi} \end{pmatrix}}^{\mathcal{M}} D^2 + \overbrace{\begin{pmatrix} \mathcal{C}_{\chi\chi} & \mathcal{C}_{\chi\psi} \\ \mathcal{C}_{\psi\chi} & \mathcal{C}_{\psi\psi} \end{pmatrix}}^{\mathcal{C}} D + \overbrace{\begin{pmatrix} \mathcal{K}_{\chi\chi} & \mathcal{K}_{\chi\psi} \\ \mathcal{K}_{\psi\chi} & \mathcal{K}_{\psi\psi} \end{pmatrix}}^{\mathcal{K}}\right] \begin{pmatrix} \chi_r \\ \psi \end{pmatrix} = \begin{pmatrix} M_{\chi_r} \\ M_\psi \end{pmatrix}$$

where $\mathcal{M}$, is the mass matrix , $\mathcal{C}$ is the damping matrix, $\mathcal{K}$ is the stiffness matrix, and D is the differential operator. Expanding $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{K}$ we have,

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{\chi\chi} & \mathcal{M}_{\chi\psi} \\ \mathcal{M}_{\psi\chi} & \mathcal{M}_{\psi\psi} \end{pmatrix} = \begin{pmatrix} T_{yy} & F'_{\lambda y} + \frac{c_f}{c_w} T_{yz} \\ F'_{\lambda y} + \frac{c_f}{c_w} T_{yz} & F'_{\lambda\lambda} + 2\frac{c_f}{c_w} F''_{\lambda z} + \frac{c_f^2}{c_w^2} T_{zz} \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{\chi\chi} & \mathcal{C}_{\chi\psi} \\ \mathcal{C}_{\psi\chi} & \mathcal{C}_{\psi\psi} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\left(H_f \cos\lambda + \frac{c_f}{c_w} H_t\right) + V\left(T_{yz}\frac{\cos\lambda}{c_w} - \frac{c_f}{c_w} m_t \overline{h}_t\right) \\ H_f \cos\lambda + \frac{c_f}{c_w} H_t & V\left(\frac{\cos\lambda}{c_w}\left(F''_{\lambda z} + \frac{c_f}{c_w} T_{zz}\right) + \frac{c_f}{c_w}\nu\right) \end{pmatrix}$$

$$\mathcal{K}=\begin{pmatrix}\mathcal{K}_{\chi\chi} & \mathcal{K}_{\chi\psi}\\ \mathcal{K}_{\psi\chi} & \mathcal{K}_{\psi\psi}\end{pmatrix}=\begin{pmatrix}-gm_t\overline{h}_t & g\nu-H_tV\frac{\cos\lambda}{c_w}-V^2\frac{\cos\lambda}{c_w}m_t\overline{h}_t\\ g\nu & -g\nu\sin\lambda+VH_f\sin\lambda\frac{\cos\lambda}{c_w}+V^2\frac{\cos\lambda}{c_w}\nu\end{pmatrix}$$

To derive one fourth order governing linear differential equation we can solve for either χ_r or ψ using equations (3.14) and (3.15). Solving for an equation in either variable will result in the same fourth order characteristic equation.

Assuming a solution of the form e^{st} results in a characteristic fourth order polynomial in s whose roots can be used to determine if the solution to the equation grows or decays in time.[14]

Alternatively, we can derive this fourth order characteristic equation by noting, just as in a linear algebra problem, that for a solution to exist for the matrix equation (3.15) that the determinant of the matrix must be zero. That is,

$$det\,(\mathcal{M}\ s^2+\mathcal{C}\ s+\mathcal{K})=0 \tag{3.16}$$

where we have replaced the differential operator D with s.

To solve for the determinant of the system we first note that that the determinant can be defined by the following,

$$det\ \mathcal{M}=\frac{1}{2}\ \mathcal{M}*\mathcal{M}$$

where $*$ is defined for any 2×2 matrices, say matrix a and matrix b, as follows,

$$a*b=(a_{11}b_{22}+a_{22}b_{11}-a_{12}b_{21}-a_{21}b_{12})$$

[14] This is discussed more in Chapter V. See Braun [1976] pp. 179 for details of the solution of this equation.

Further noting $a * b = b * a$ we use this definition to define the determinant of the equation (3.16) as follows,

$$\frac{1}{2}(\mathcal{M}\ s^2 + \mathcal{C}\ s + \mathcal{K}) * (\mathcal{M}\ s^2 + \mathcal{C}\ s + \mathcal{K}) =$$
$$\frac{1}{2}(\mathcal{M} * \mathcal{M})s^4 + \frac{1}{2}(\mathcal{M} * \mathcal{C})s^3 + \frac{1}{2}(\mathcal{M} * \mathcal{K})s^2$$
$$+ \frac{1}{2}(\mathcal{C} * \mathcal{M})s^3 + \frac{1}{2}(\mathcal{C} * \mathcal{C})s^2 + \frac{1}{2}(\mathcal{C} * \mathcal{K})s$$
$$+ \frac{1}{2}(\mathcal{K} * \mathcal{M})s^2 + \frac{1}{2}(\mathcal{K} * \mathcal{C})s + \frac{1}{2}(\mathcal{K} * \mathcal{K}) =$$
$$det\ \mathcal{M}\ s^4 + (\mathcal{M} * \mathcal{C})s^3 + (\mathcal{M} * \mathcal{K} + det\ \mathcal{C})s^2 + (\mathcal{K} * \mathcal{C})s + det\ \mathcal{K} = 0 \quad (3.16a)$$

We define for our use later in Chapter V the coefficients of the fourth order polynomial as $A, B, C, D, andE$,

$$As^4 + Bs^3 + Cs^2 + Ds + E = 0 \qquad (3.17)$$

Thus,

$$A = \frac{1}{2}\ \mathcal{M} * \mathcal{M} = det\ \mathcal{M} \qquad (3.18a)$$

$$B = \mathcal{M} * \mathcal{C} \qquad (3.18b)$$

$$C = \mathcal{M} * \mathcal{K} + det\ \mathcal{C} \qquad (3.18c)$$

$$D = \mathcal{K} * \mathcal{C} \qquad (3.18d)$$

$$E = \frac{1}{2}\ \mathcal{K} * \mathcal{K} = det\ \mathcal{K} \qquad (3.18e)$$

Conclusion

Once the equations of motions have been derived they can be used in various type analyses. Among them are:

1) Given arbitrary χ_r and ψ we can calculate the required moments M_{χ_r} and M_ψ.

2) If $M_{\chi_r} = 0$ we can solve for the behavior of χ_r for a prescribed ψ, i.e. we can define a controller $\psi(\chi_r)$ and analyze stability.

3) Given that $M_{\chi_r} = 0$, we can solve for M_ψ and ψ.

4) Given $M_{\chi_r} = 0$ and $M_\psi = 0$, we can solve for the equation of motion and analyze the bicycle's self stability.[15]

5) Passive mechanisms such as gyroscopes, springs, and dampers can be easily added to the equations of motion to analyze their effect on the bicycle.

This thesis focuses on the passive rider 'self stability' of the bicycle. For this reason we shall next investigate the equations (3.14), (3.15), and (3.16) when $\mathcal{M}_{\chi_r}$ and $\mathcal{M}_\psi$ are equal to zero.

[15] This is done in the next chapter.

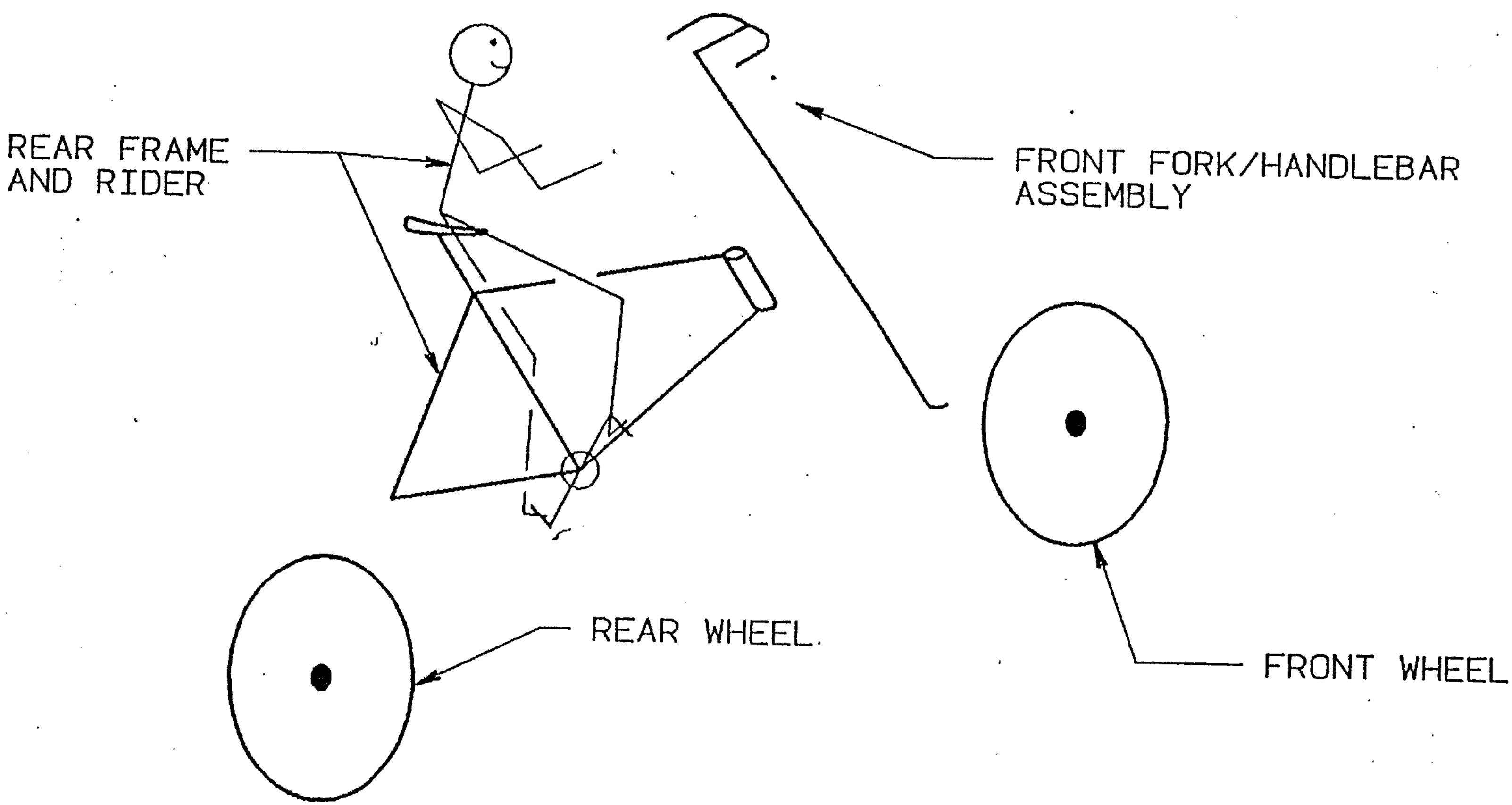


FIGURE 3.1

Four rigid bodies that constitute the Basic bicycle model.

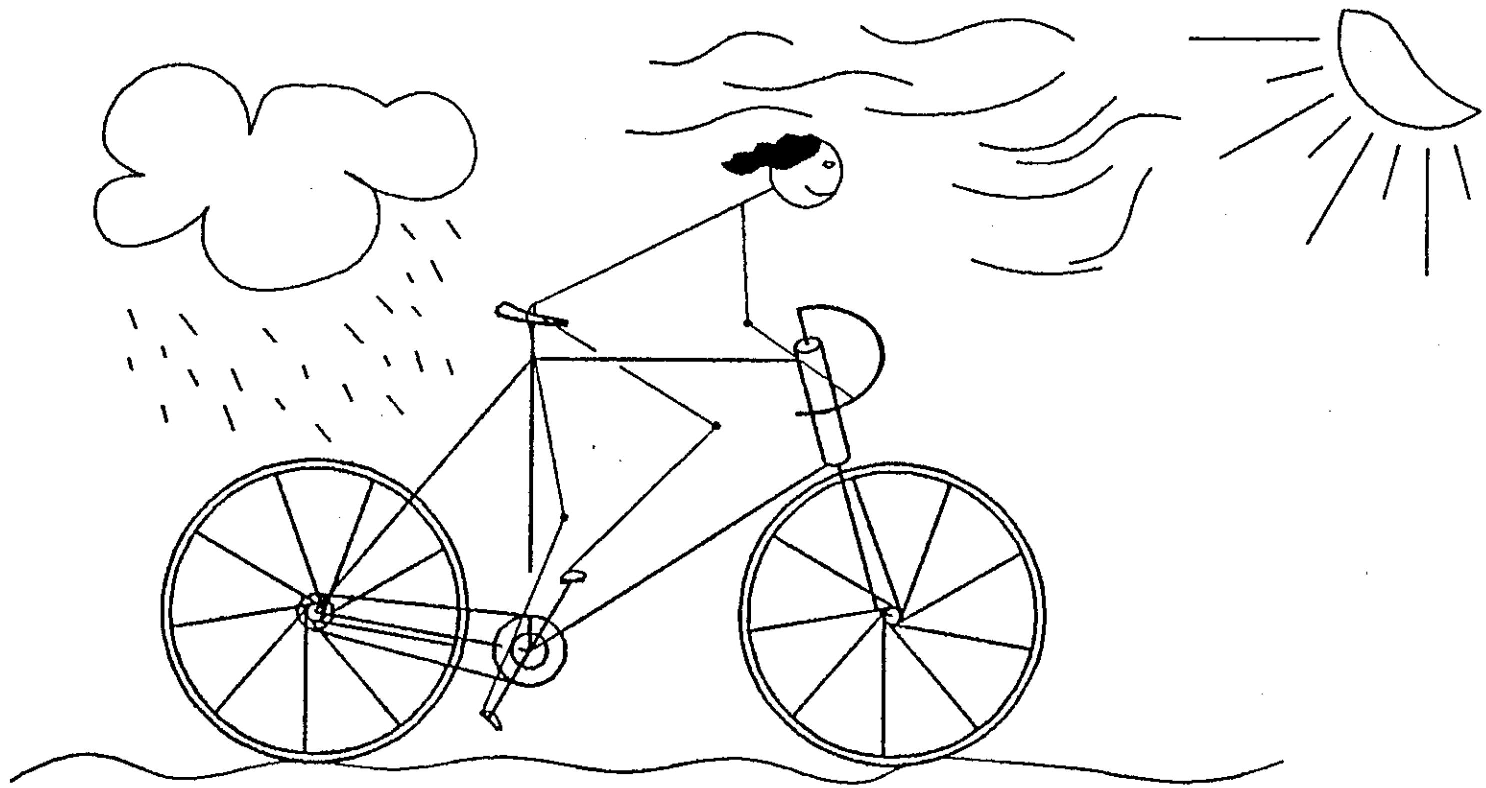


FIGURE 3.2a

A real life bicycle and rider.

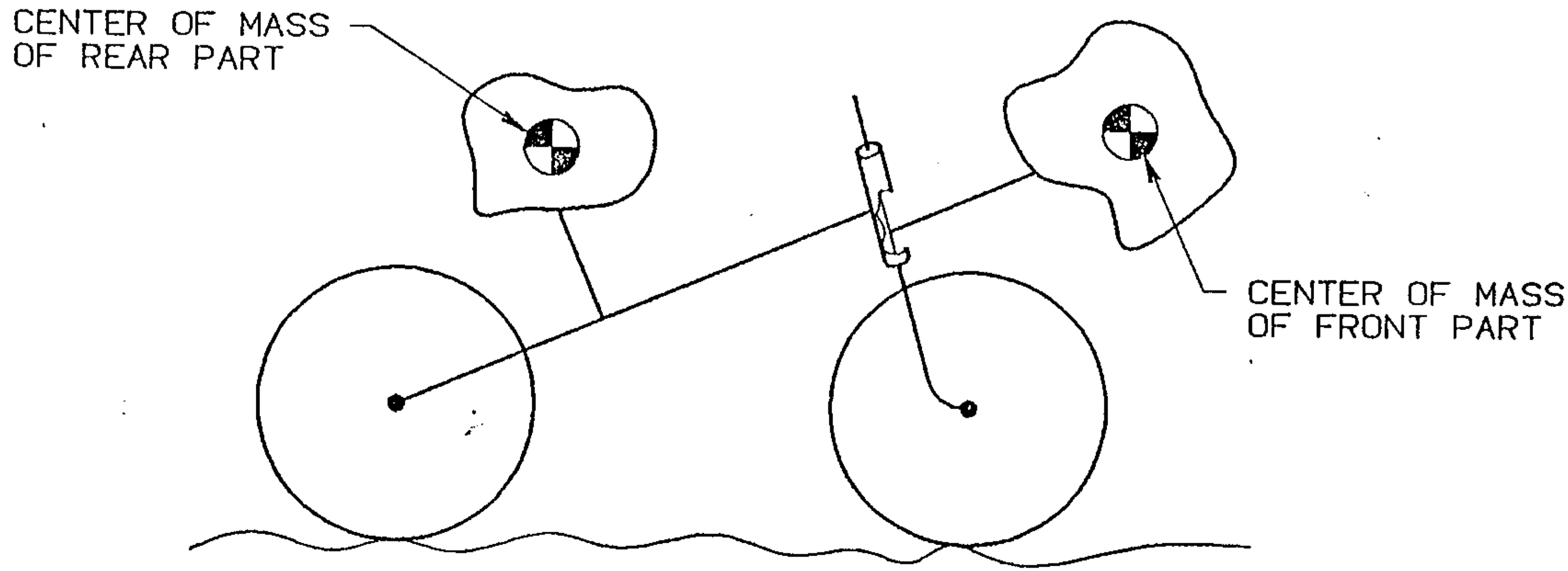


FIGURE 3.2b

Assume rigid bodies and no friction, and we get two "linked" rigid bodies on wheels, front part and rear part (front and rear parts include front and rear wheels respectively).

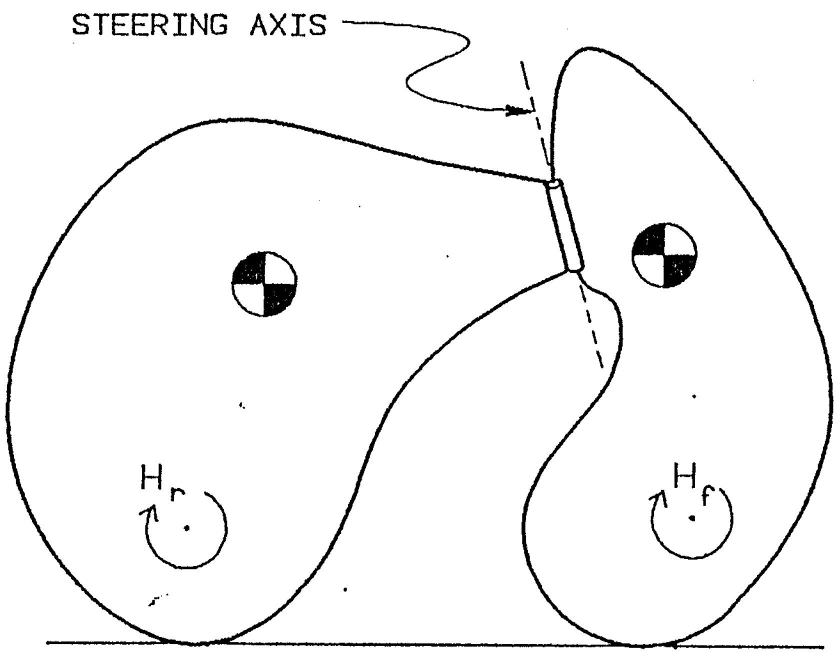


FIGURE 3.2c

Further assume the ground to be flat and wheels are thin disks. Replace the wheels as fast spinning gyrostats, where H_r and H_f are the angular momentum of the rear and front wheel, respectively.

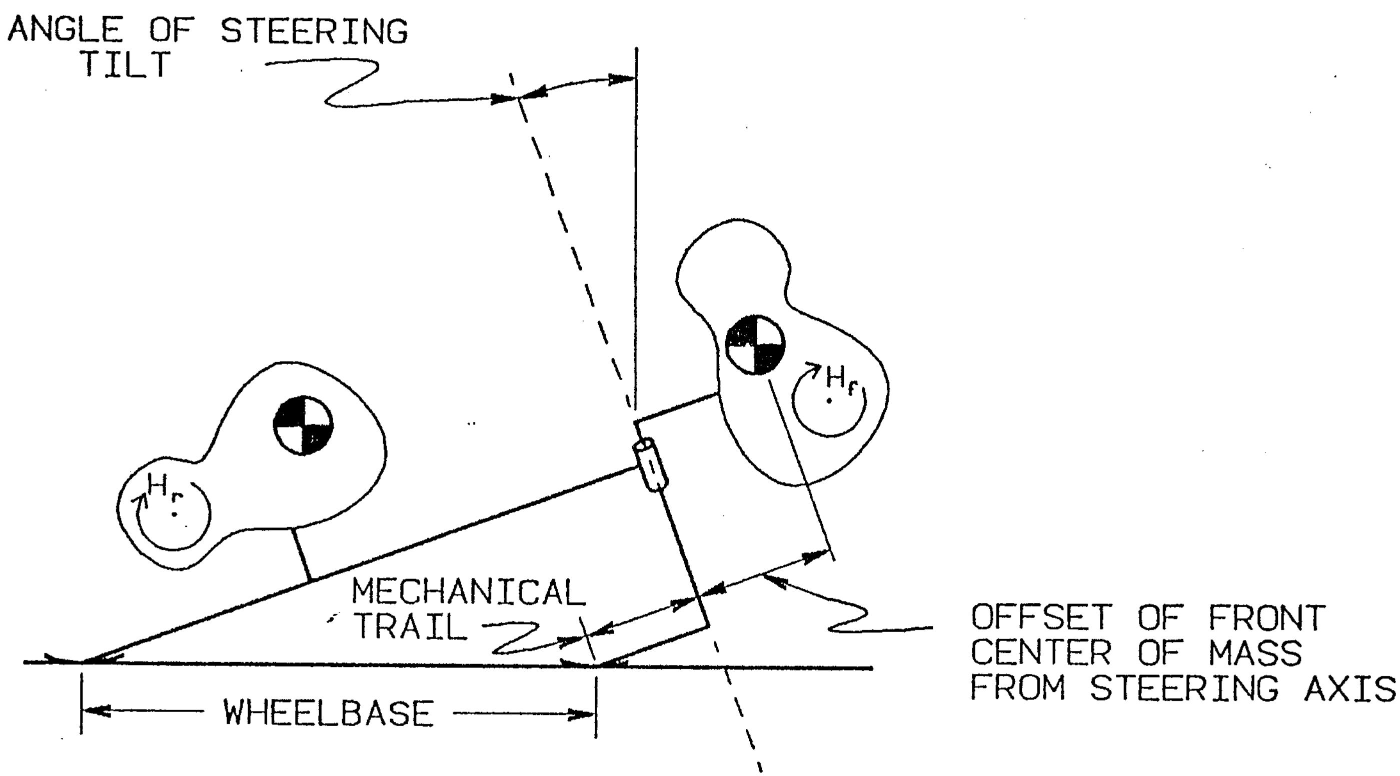


FIGURE 3.2d

Reduce the thin disks to tiny ice skate blades. Label pertinent geometry.

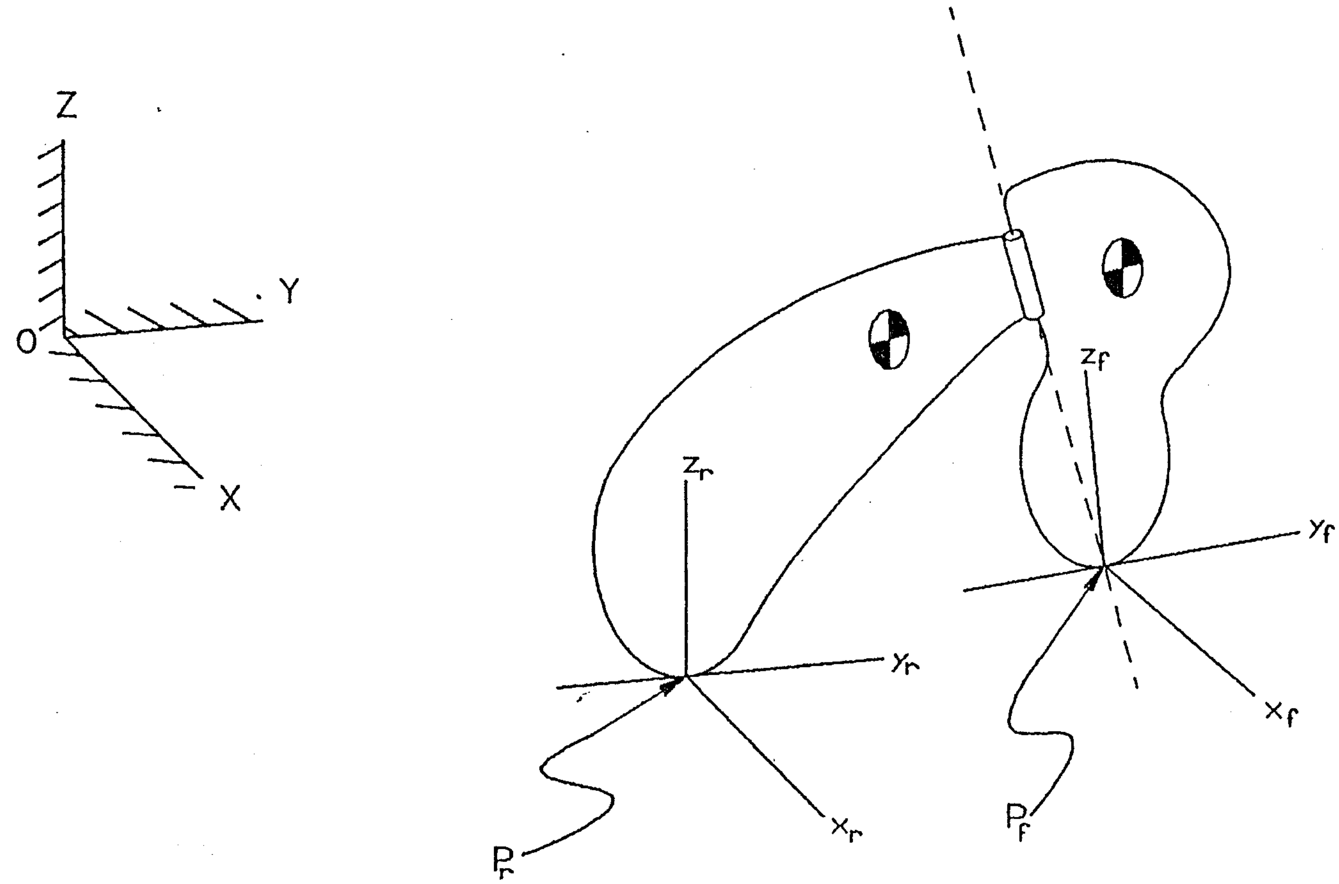


FIGURE 3.3
Inertial reference frame XYZ and moving reference frames $x_r\,y_r\,z_r$ and $x_f\,y_f\,z_f$.

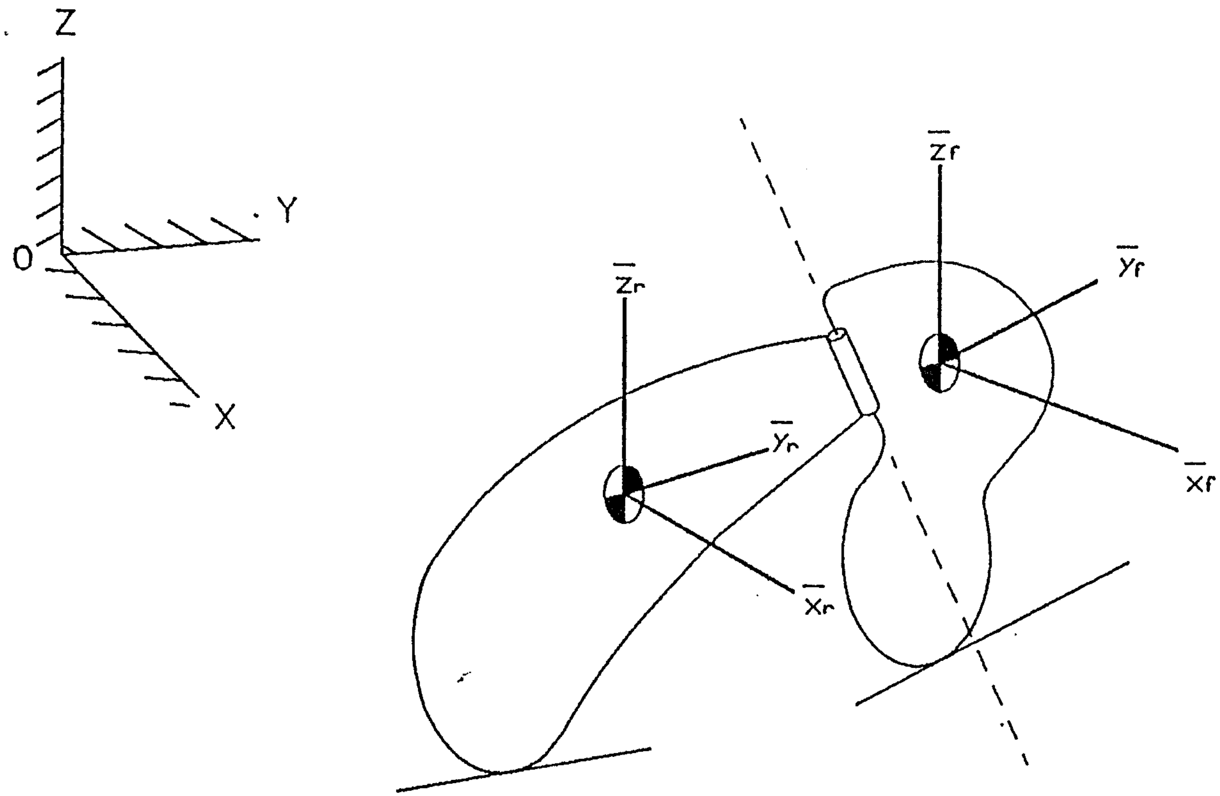


FIGURE 3.4

Inertial reference frame XYZ and moving reference frames $\overline{x}_r\,\overline{y}_r\,\overline{z}_r$ and $\overline{x}_f\,\overline{y}_f\,\overline{z}_f$.

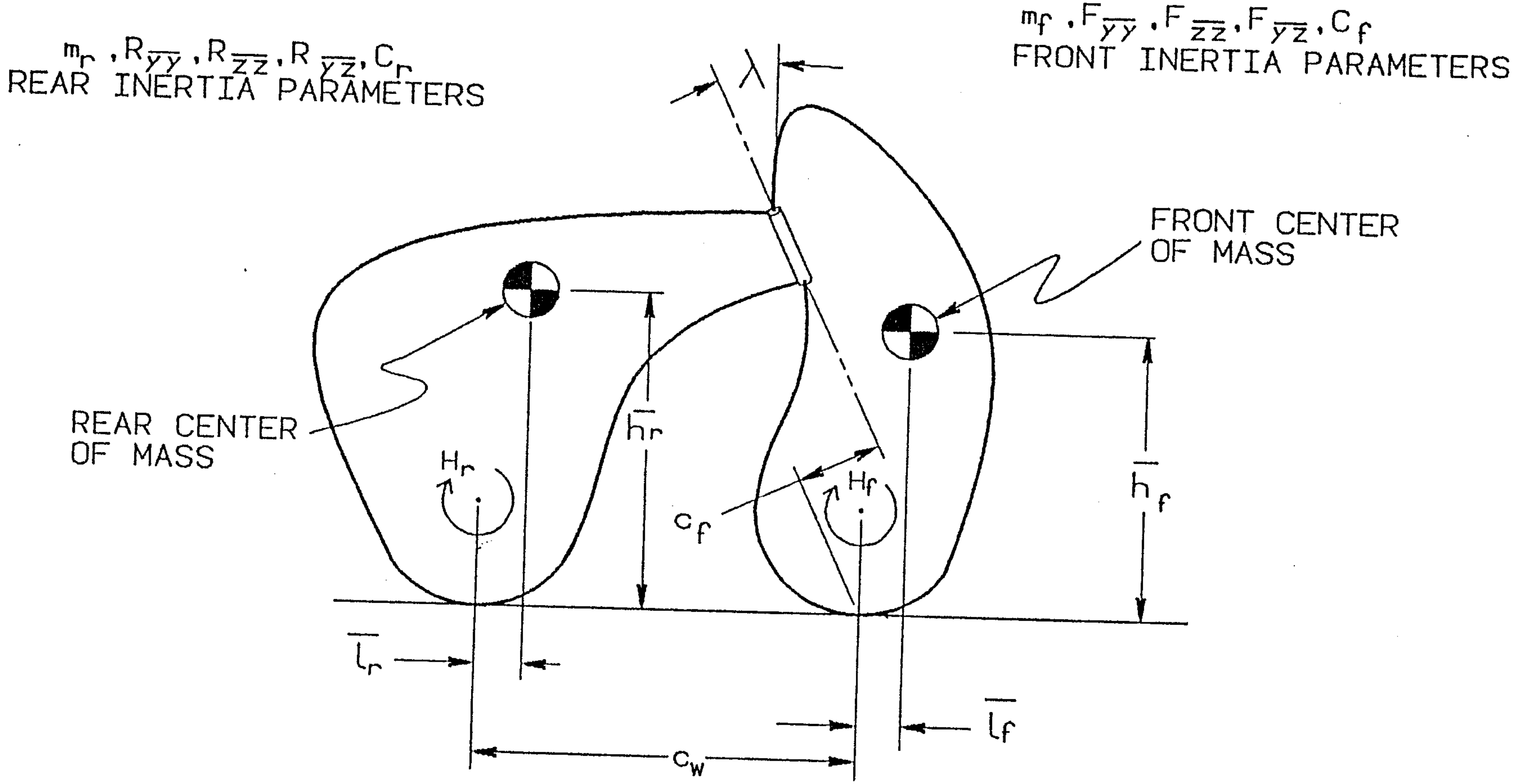


FIGURE 3.5a

Geometric and inertia design parameters. Angles and distances shown in their positive configuration.

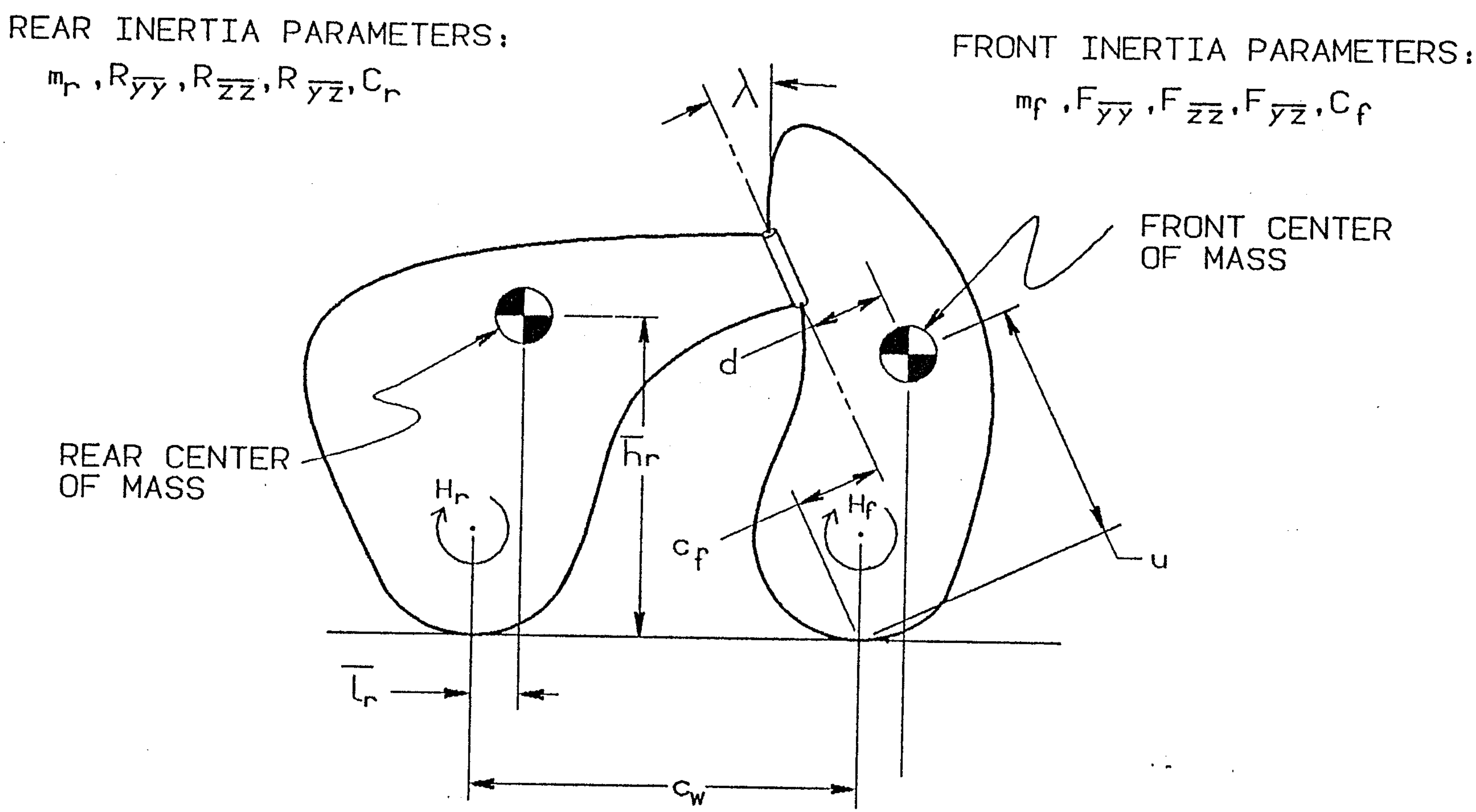


FIGURE 3.5b

Geometric and inertia design parameters. Angles and distances shown in their positive configuration.

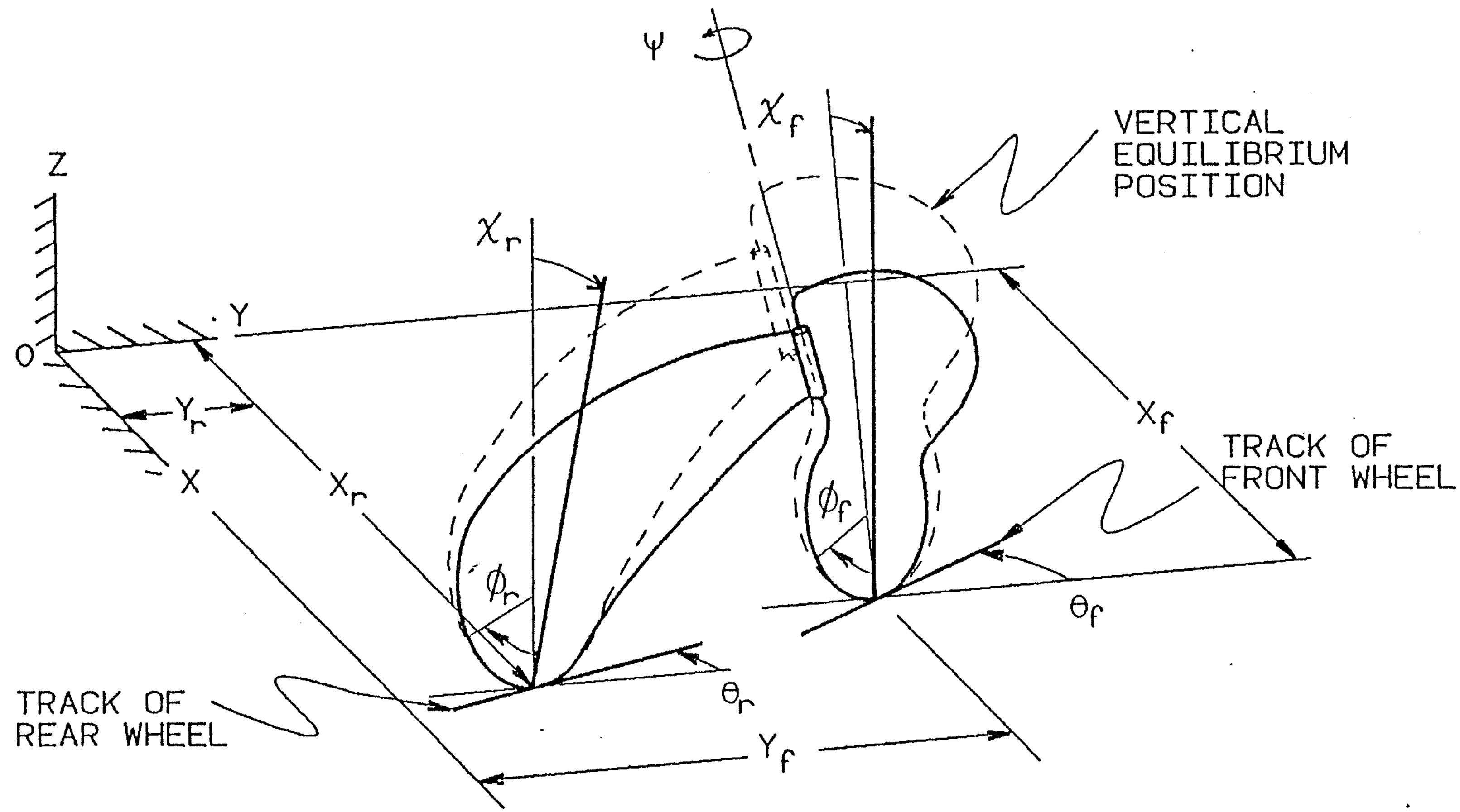


FIGURE 3.6

Generalized coordinates $X_r, Y_r, \theta_r, \chi_r, \Psi, \phi_r, \phi_f$ and auxiliary variable, $X_f, Y_f, \theta_f, \chi_f$.

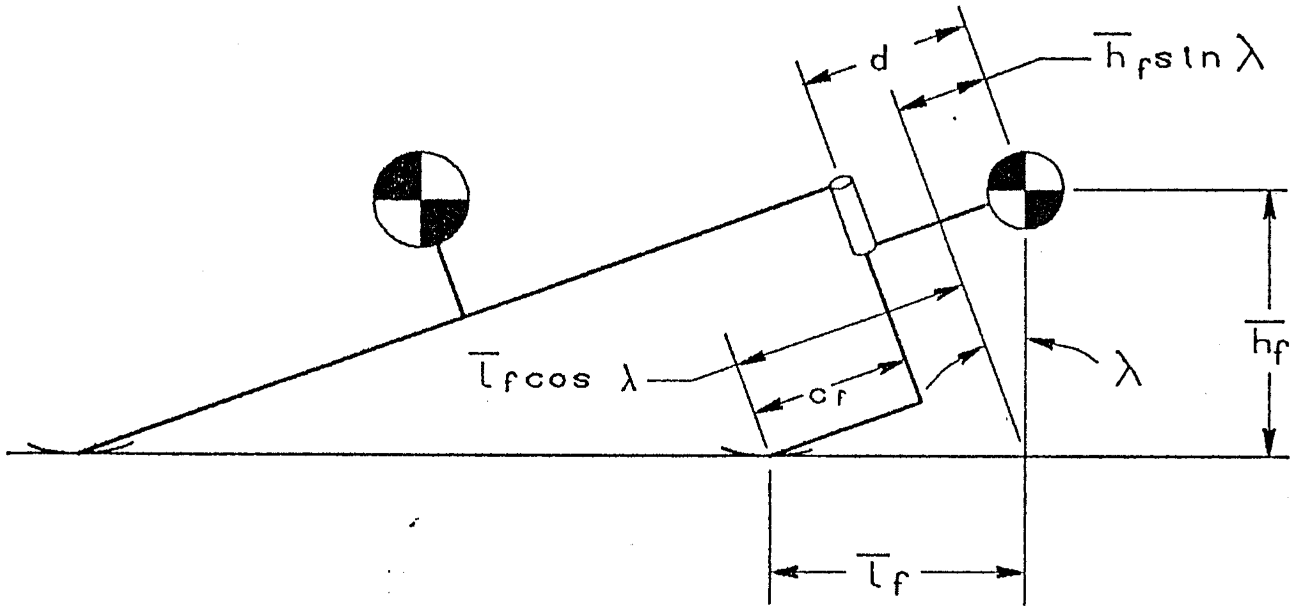


FIGURE 3.7

Definition of the perpendicular distance from the front center of mass to the steering axis, d.

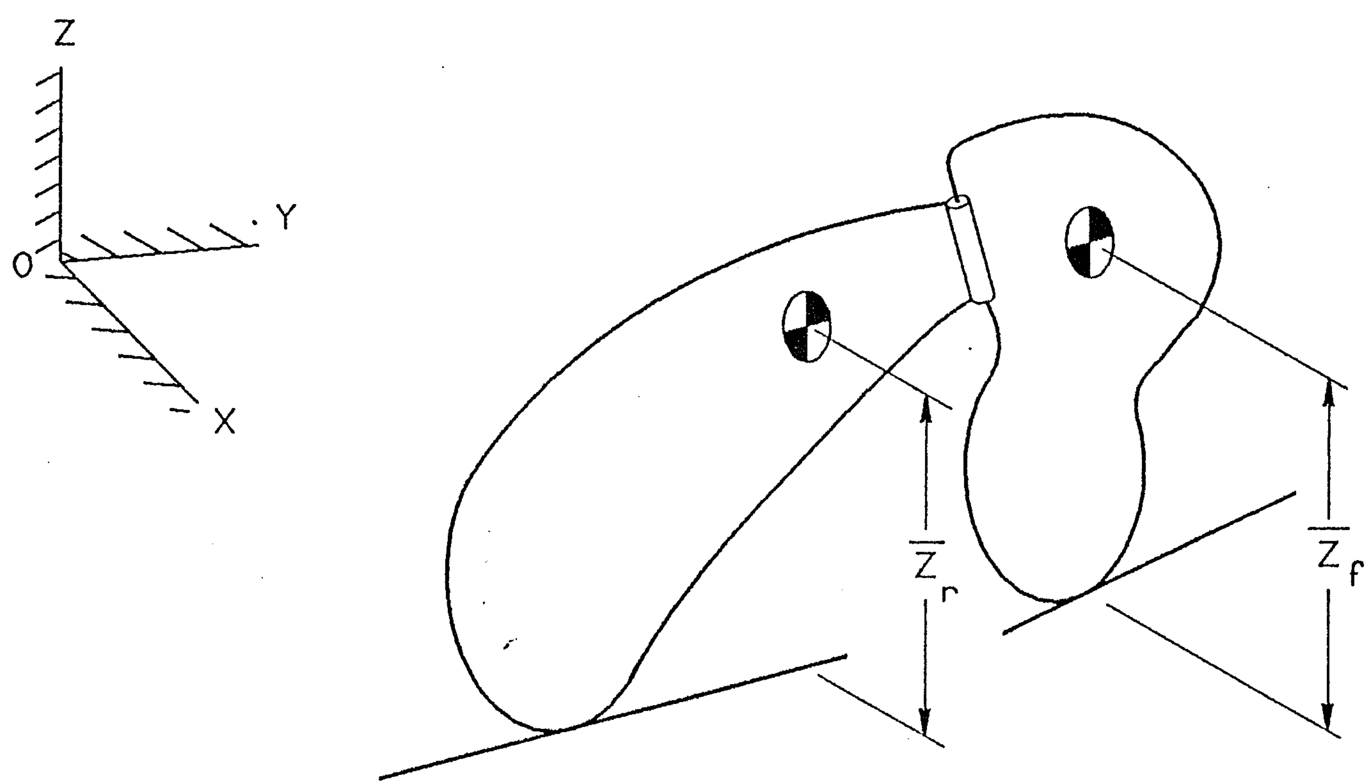


FIGURE 3.8

Definition of the vertical distance to the rear and front center of mass $\overline{Z}_r$ and $\overline{Z}_f$.

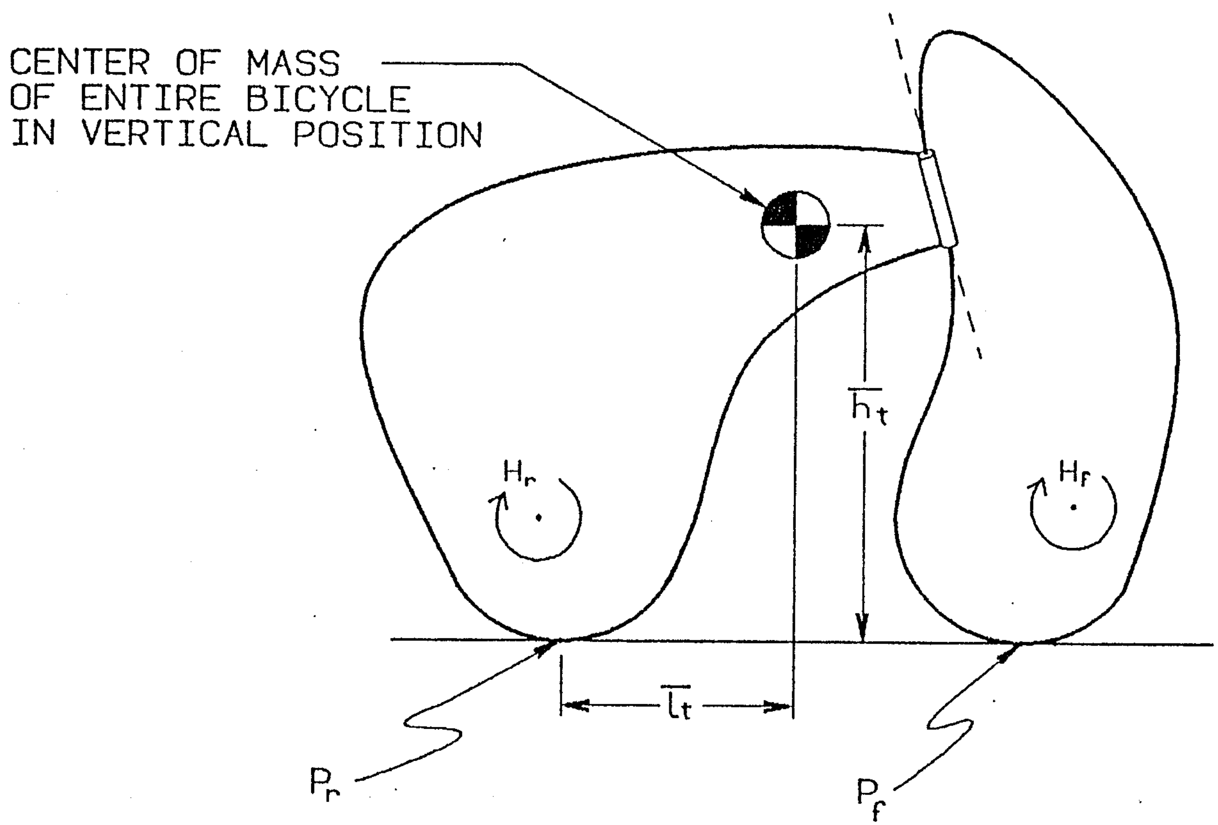


FIGURE 3.9

Location of the center of mass of entire bicycle relative to the rear contact point, $\overline{l}_t$ and $\overline{h}_t$.

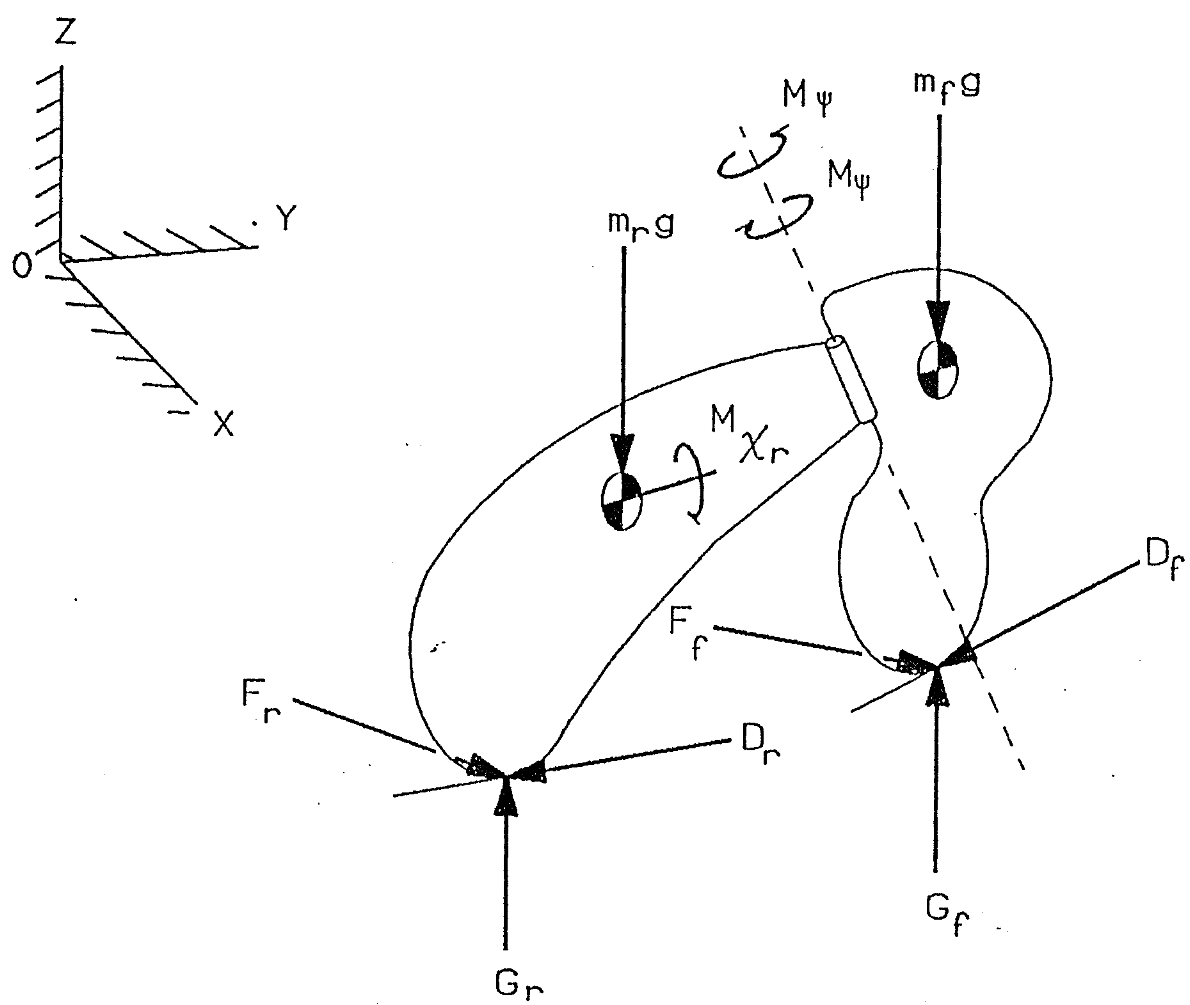
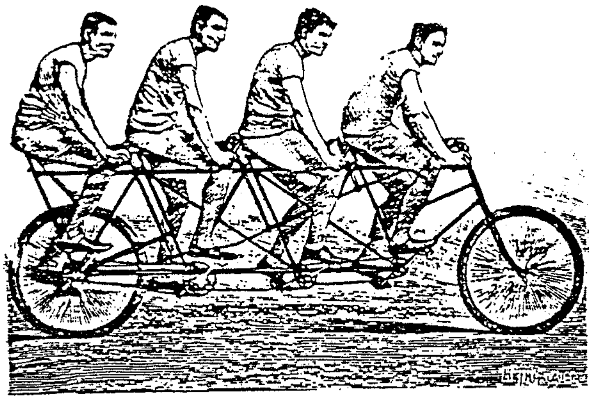


FIGURE 3.10
Free body diagram of the basic bicycle model.



IV. CHRONOLOGICAL COMPARISON OF THE LINEARIZED EQUATIONS OF MOTION FOR THE BASIC BICYCLE MODEL

Introduction

In Chapter III of this thesis the linearized equations of motion for a Basic bicycle model were presented. This chapter chronologically compares other authors' linearized equations of motions for a bicycle to those we derived in Chapter III. When it was not possible to directly compare equations, we have tried to simplify other authors' equations to represent the linearized equations of the Basic bicycle model, and/or we simplified the equations from Chapter III to represent the model studied by the particular author. In some cases neither approach was possible due to the complexity of the other equations. For these cases we have given a brief description of the their equations, and when possible have commented on the likelihood of correctness.[1]

The purpose of comparison was to gain confidence in our derived equations and to see which studies could legitimately be used without modifications. Knowing if other authors' equations agree with ours also gives us a basis for evaluating their conclusions. It is as a result of the numerous comparisons given in this chapter, that we developed and tailored what we feel is the easiest way to correctly derive the linearized equations of motion, as presented in Chapter III.

The equations of motion of a bicycle have mainly been derived either from Lagrange's equations, or using Newton's Laws on the individual rigid bodies which make up a bicycle. Chapter III describes the derivation of the linearized equations of motion using Lagrange's equations with nonholonomic constraints. Döhring [1955] derived an equivalent set of linearized equations using Newton's Laws. Weir [1972] gives a four degree of freedom Newtonian derivation using vector notation. To our knowledge we are the first to derive correct linearized equations of motion for a fully general Basic bicycle model with Lagrangian methods. It seems that using Lagrange's equations (or at least the concept of generalized coordinates) is a simpler approach, because it eliminates the requirement of solving complicated simultaneous equations representing the force and moment balance on the 4 rigid bodies that make up the Basic bicycle model.[2] We mention however, that the Lagrangian approach suffers because many students are not exposed to Lagrangian

[1] This chapter includes some of the conversion notations required to make our comparisons.

[2] See for example Roland [1971].

dynamics at the undergraduate level, especially for systems with nonholonomic constraints. In this sense, the Basic bicycle model could be used to introduce the subject, having general appeal and pedagogical meaning.

On the whole, the previous literature concerning the equations of motion suffers from three major flaws:

1) Some derivations seem impenetrable. This results from leaving out steps, from choosing notation which is not well suited to the job, from using roundabout procedures when more direct ones are possible, from not simplifying the results afterwards, and from not explaining their physical significance. The resulting equations are often far too complicated to use, except numerically. Some equations are so long that it takes several pages just to define the coefficients. Most of these studies do not enhance the reader's understanding of bicycle motion.

2) Few or no comparisons were made to works by previous authors, so their correctness was not known, and earlier results were 'lost'. Only one author explicitly stated that he had compared his equations to a previous author's.

3) The models used by some authors have ignored major stability-related design parameters. Some lack steering axis tilt, have only point masses, or make other assumptions restricting distribution and location of mass.

What follows is a chronological comparison of all papers we have found in which equations of motion for bicycles (or motorcycles) are presented. Because we feel that failure to compare to others' equations (especially when they are cited as a reference) is inexcusable, we have noted each author's comments on previous works.

In our comparisons, we found that in fewer than half of the papers do the equations of motion resemble those derived in Chapter III. In fact, of the papers discussed, we found that only two derived fully general and perfectly correct results (one of these was later employed by another investigator). Several more were either a little less general or had minor errors which an alert reader might catch. A number of others were too complicated to check in full (but some of them raised some questions we could not answer). Finally, several are just plain wrong.

Results of Chronological Comparison of Linearized Equations of Motion[3]

Whipple, 1899

The first to formally derive a fully general and scholarly set of equations for the Basic bicycle model was Whipple in 1899. He treats the front and the rear parts symmetrically throughout the derivation. He derived nonlinear governing equations of motions for a Basic bicycle model with an active (leaning) rigid rider, and then linearized about the vertical equilibrium configuration. His equations of motion can be found as eq. XIV, eq. XV, and eq. XIII in his paper on pp. 321-323, but not all terms are defined. The equation are restated more clearly and explicitly in matrix form on p. 326. We also note that the figure defining some of his variables is at the end of the bound volume containing his paper.

[3] Comparisons to works by Whipple [1899], Carvallo [1901], Sommerfeld and Klein [1903], and Döhring [1955] were performed mainly by Dr. Jim Papadopoulos, whose results are summarized here. Some of his understanding and commentary on other comparisons are contained in other parts of this chapter.

It is most convenient to compare Whipple to Döhring [1955] since similar axis orientation is used. The equations on page 326 are in the form of the 3×3 matrix which operates on his variables ϕ, ϕ', τ, where λ is the derivative operator $\frac{d}{dt}$. There are a few evident typos: the first term of the second row should have λ^2 not λ_2; and the third column second row should have $W\gamma$, not $W'\gamma$) .

We found his notation to be more difficult to understand than most and therefore give some details about the comparison. In his notation,

$$\phi = \frac{\psi}{\sin\theta}$$

and

$$\phi' = \frac{\psi}{\sin\theta} - Q$$

where ψ is the lean of the rear frame and Q is the steer angle. (In our notation the lean of the rear frame is χ_r and ψ is the steer angle.) The last equation in his matrix defines τ as a function of ϕ and ϕ' and allows one to eliminate τ from the first two equations of the matrix. Doing so, one finds the first equation is in complete agreement with our lean equation when ϕ and ϕ' are written in terms of ψ and Q. The second equation of the matrix, when it is corrected and then multiplied by $\frac{\mu\mu'}{b\cos\theta}$, we find agrees completely with Döhring's [1955] equation (31). As is explained below, Döhring's equation (31) is a linear combination of our lean and steer equation and thus Whipple's linearized equations are in complete agreement with those presented in Chapter III. His work, which is as sophisticated as almost any later investigation, was evidently done for his degree from Trinity College, Cambridge University.

Overall, the definitions of Whipple's variables are difficult to decipher and make his paper difficult to read, but his equations appear to be rigorously derived and are fully general when compared to those given in Chapter III. Whipple is critical of McGaw's [1898] study of tricycles, and Bourlet's [pre–1896] study of bicycles, neither of which have we read.

Carvallo, 1901

Carvallo [1901] wrote 300 generally lucid pages on the stability of monocycles (rider inside a single wheel) and bicycles. Only the second part of the monograph, which won a prestigious prize, concerns us. In it he modifies Lagrange dynamics to deal with rolling hoops and bicycles (we were not able to tell if his method is a different way of dealing with nonholonomic constraints). We are concerned primarily with section V on no-hands stability. The equations where each term was derived are laid out on pp. 100-101, and restated in condensed form on p. 103. The equations are exactly analogous to ours, one for lean and one for steer.

Although we could not find where Carvallo said this, it appears that his bicycle has two identical heavy wheels, the rider and frame are considered a single unit, and the mass of the front assembly is at the center of the front wheel and its inertia properties are those of the wheel. (This is not an unreasonable idealization if the handlebars are not heavy and are positioned *on* the steering axis as was common in designs of that day. Technically, for such a design the mass of the handlebars and straight part of fork can then be considered part of the rear frame.)

We find that Carvallo's equations (for a bicycle with massless forks and handlebars) agree exactly with ours. Most quantities are defined in the text, but the reader should note that the wheel inertias are defined relative to their ground contact, i.e. C_1 is for spin about a diameter, A_1 is for lean (i.e., $A_1 = C_1 + \mu_1 R^2$), B_1 is for rolling about the contact point (i.e., $B_1 = I_p + \mu_1 R^2$). $S = \frac{V}{R}$ is the wheel rotation rate. Carvallo makes no reference to other works, which is not surprising as his research was evidently performed in 1898.

Sommerfeld and Klein, 1903

Sommerfeld and Klein (S & K) in 1903 derived the linearized equations of motions for the Basic bicycle model having all the mass and inertia of the front assembly in the front wheel (similar to Carvallo). Somewhat similar to Whipple [1899], they used a Newtonian analysis of the front and rear assembly, and treated the two parts as two trailers attached to the steering axis, deriving the linearized equations of motions using axes parallel to the steering axis. S & K refer to Whipple [1899] and Carvallo [1901] but do not say whether their equations agree.

Their equations are most easily compared to Döhring's [1955], and are found to be a correct subset of his. It is possible that S & K's slight simplification(s) to the model were due to their main interest in determining what effect the wheels as gyroscopes had on the stability (since the article is a chapter in their massive work on gyroscopes). They are critical of Bourlet [1898] (whose book we have not read).

Bower, 1915

Bower in 1915, without reference to any previous bicycle work, derived the linearized equations of motion for a simplified Basic bicycle model at the end of an article mainly concerning the gyroscopic effects of the engine and wheels on steady turns. His model consists of a rear frame with its center of mass above the rear contact point, having polar inertia $R_{\overline{zz}}$ provided by two point masses, one ahead and one behind the center of mass. Two smaller masses at the same height are attached to the front assembly. Wheel inertia and caster trail are also included, but the steering axis is restricted to be vertical.

Instead of providing two second order equations for his model, he presents the governing 4th order linear differential equation (eq. (19) in his analysis), which is not convenient for comparison. The e coefficient, given as equation (24) in his paper, is comparable to the determinant of the $\mathcal{K}$ matrix presented in our Chapter III. Comparison indicates that Bower's equations must be missing the $g\nu$ term in the $\mathcal{K}_{\chi\psi}$ coefficient of the lean equation for his simple model, which confirms that his equations lack some of the effects of trail has on the bicycle. No comparison was made to Bower's coefficients A-D for his simplified bicycle model, but casual observation indicates they also lack terms.

Looking back at his derivation it appears that his ϕ is our $-\psi$, and his θ is our χ_r. His eqs. (15) and (16) may be added to eliminate the internal reaction P, thus leading to a lean equation. However, (a) he has ignored product of inertia terms (relative to the wheel contacts) which should appear multiplying his $\ddot{\psi}_1$, $\ddot{\psi}_2$; this is correct for the rear part of his simplified model, but not for the front unless trail vanishes. Also, (b) he has left out the lateral offset of the front and rear mass center from the track line due to steer angle; this too is correct for the rear part of his simplified bicycle but not for the front unless trail vanishes. (It also appears that he should have included a vertical reaction force at the steering bearing, though this would cancel when (15) and (16) are added.) Finally (c) his centrifugal forces (such a f_1 are in error because he assumes a steady curve due to steer angle divided by a finite wheelbase, whereas in fact even with an infinite wheelbase the *rate* of steer can produce path curvature of the front wheel and with nonzero trail the rate of steer also affects the yaw rate of the rear wheel. Based on these observations, it seems likely that his lean equation could apply correctly to his simplified model only when the trail is zero.

We believe the steer equation could be is formed by adding $(1+\frac{c}{l})(eq.\ 17)+(\frac{c}{l})(eq.\ 18)$ to eliminate P (the term multiplied by his $\ddot{\theta}$), but we have not checked this in detail.

Pearsall, 1922

In 1922 Pearsall, with the stated intention of extending Bower's [1915] ideas and discovering the cause of "speedmans wobble," derived a set of equations for a bicycle model somewhat similar to the Basic bicycle model presented in Chapter III. He never states precisely whether his model is restricted in any way, but for example, his equations don't include any product of inertia terms, so they are probably not general.

His technique for deriving the equations of motion was to first linearize the equations of motion of a rolling hoop and then "add on" the trailer effects due to the remaining parts of the bicycle using fairly casual arguments. While his brief verbal justifications sound valid, in fact almost no terms in the equations are exactly correct. We did not make the effort to trace his errors, but note that there may have been a major mistake in the kinematical treatment (which is not spelled out very explicitly): the headings γ and θ of the rear and front assemblies are defined relative to the track line, but then they appear to be treated as coordinates relative to inertial space in the equations.

We compared his equation (4) to our steer equation and his equation (5) to our lean equation, and found that his equations differ significantly in almost every term when compared to those presented in Chapter III. His equations would also disagree with Bower's if simplified for Bower's model.

Pearsall does not say if he compared his equations to Bower's, and he does not refer to any other works.

Timoshenko and Young, 1948

In this textbook on advanced dynamics, Timoshenko and Young derived a nonlinear (large-angle) lean equation for a simplified Basic bicycle model having only a point mass in the rear part of the bicycle, and a steer angle controlled by the rider. Their model neglects wheel inertias, steering axis tilt, trail and front-mass offset from the steering axis. When linearized, we find this lean equation agrees with our lean equation simplified for an equivalent configuration.

Döhring, 1955

In 1955, in order to more generally analyze the stability of motorcycles and motorscooters, Döhring extended Sommerfeld and Klein's (S & K) [1903] linearized equations for the Basic bicycle model by allowing the mass distribution of the front assembly to be fully general. Just as S & K did, Döhring used Newton's Laws to derive the equations of motion in linearized form, rather than linearizing from nonlinear equations as Whipple had.

Döhring's final equations were found to be in exact agreement with those derived in Chapter III. In order to compare his equations to ours we made the following substitutions in his equations (29) and (30) of his [1955] paper,

$$\psi = \gamma \cos \sigma$$

$$\theta_1 = \theta_2 - \gamma \sin \sigma$$

where γ is steer angle (our ψ) and θ_2 is lean angle (our χ_r). When these substitutions are made Döhring's equation (30) is exactly our lean equation. Our steer equation results from the linear combination of Döhring's equation (31) and (30). Using Döhring's notation this combination is as follows:

$$\frac{(eq.\,31)}{l} + \frac{c_1 \sin \sigma (eq.\,30)}{l} = -M_d = our \;\; M_\psi$$

Although not rigorous in how his linearizations are made, Döhring's derivation was fairly easy to follow, and offers a good physical description of the variables and equations of motion. Döhring refers to S & K, but never states explicitly how his equations compare.

Collins, 1963

In his 1963 University of Wisconsin Ph.D. dissertation R. N. Collins, working on a project supported by Harley Davidson Motor Company, studied a Basic bicycle model with the addition of a driving force on the rear tire and an explicit force for aerodynamic drag applied to the front fork/handlebar assembly. He derived the equations of motion using Euler's equations (Newton's Laws) for the 4 rigid bodies of the Basic bicycle model.

Collins derives nonlinear velocity and acceleration expressions for the rear and front center of mass first (see pages 19 and 20 of his dissertation), and then linearizes about the vertical equilibrium position, before deriving the linearized equations of motion. By writing the drive force and aerodynamic drag force as a function of the square of the forward velocity of the motorcycle (see p. 12 in his dissertation), he alters the vertical contact forces on the front and rear wheels. By making the assumptions of no slip angle and constant velocity he has only two degrees of freedom for his model and he is therefore able to write the linearized governing equations as two coupled second order ordinary differential equations in the lean and steer angles (see p. 76 eq. (5.1) and eq. (5.2) in his dissertation). The final equations are complicated in appearance and include over 30 quantities defined in terms of motorcycle parameters. (These quantities often include previously defined quantities, which further complicates understanding of the equations.)

His equation (5.1) is not exactly the steer equation, and his equation (5.2) is not exactly the lean equation. However, if we transfer all the terms to the left hand side, and form the combination,

$$\sin\alpha[eq.\,(5.1)] + h_2[eq.(5.2)] = [equation\ with\ no\ \dot{\phi}\ and\ no\ M_3] = 0,$$

the result appears to be the lean equation. That is, in our notation the coefficients $\mathcal{M}_{\chi\chi}$, $\mathcal{C}_{\chi\chi}$ (which is zero), $\mathcal{K}_{\chi\chi}$ are all in agreement with those presented in Chapter III. The steering moment M_3, our equivalent M_ψ, also drops out of the equation as it should. So while the task of multiple substitution was tedious and prevented us from completely comparison of the lean equation, or even from determining what combination of his equations ought to give our steer equation, it may be that Collins' resulting equations are correct.

The only potential flaw to come to light is that Collins' equivalent to our $\mathcal{C}_{\chi\psi}$ term, namely

$$-(\sin\alpha K_{21} + h_2 K_{31}),$$

should probably include the angular momentum of both wheels. However, this expression appears to contain only the front moment of inertia I_1', not I_2'.

Collins refers to the works of Sommerfeld and Klein [1903], Bower [1915], Pearsall [1922], and Döhring [1955], but never compares his equations to theirs (nor to those of Whipple [1899] or Carvallo [1901], who were cited by S & K).

Singh, 1964

One year later, working on the same project sponsored by Harley Davidson Motor Company, D. V. Singh's Ph.D. dissertation added tire side slip to Collins' model. For reasons not stated, Singh rederived the equations of motion in a notation similar to Collins, with just a few modifications for tire side slip.

Singh's final equations are (6.11-d) and (6.12-d) on p. 74 of his dissertation. These equations were judged too impenetrable to compare to those in Chapter III, because the coefficients are defined in terms of secondary quantities, which in turn are defined as functions of physical parameters. However, on p.49 he assumes that the tire corning forces (tire side slip) are proportional to the steer angle, which is only true for steady turns. Hence, we judge at least his treatment of side-slip (eq. 4.31) to be incorrect, though if sideslip is prevented we can't say whether or not his equations are correct.

Surprisingly, it was noted by casual review of Singh's and Collins' theses that disagreement exists in their expression for the velocity of the rear center of mass of the vehicle. This can be found on page p. 52 of Singh's dissertation eq. (4.40a-c) and p. 19 of Collin's dissertation eq. (2.13a-c). Equation 4.40(a) of Singh's dissertation has an extra term compared to 2.13(a) of Collins, and some signs appear to be different in subsequent equations, although the coordinate axes chosen in both treatments seem to be equivalent.

Though he refers to Collins and to Collins' references, Singh does not compare his equations to anyone.

Neĭmark and Fufaev, 1967

In 1967 Neĭmark and Fufaev (N & F) with a brief reference to Döhring [1955] derived equations of motion of the bicycle as a classical example of a nonholonomic system. In their derivation they use Lagrange's equations with nonholonomic constraints for the path of the wheels and obtain the linearized equations of motion for the Basic bicycle model. It is their derivation that our Chapter III mainly follows.

The equations in their book which represent the relations between the auxiliary variables and generalized coordinates, linearized rolling constraints, kinetic energy for the rear and front part of the bicycle, potential energy of the bicycle, and equations of motion, can be found starting on p. 334 as eq. (2.10), eq. (2.15), eq. (2.26) and eq. (2.29), eq. (2.30), and eq. (2.37-38), respectively.

As mentioned in Chapter III an error is made in their formula for potential energy eq. (2.30). (The correct potential energy to second order is found in section 4 our of Appendix A.) This error results in the incorrect coefficients a_4, b_3, and b_4 in eqs. (2.37-38), where gm_2d should be replaced by $g\left(m_2d + \frac{c_1}{c}l\right)$. In addition to these corrections the reader should note that a typographical error occurs in the b_2 term on p. 344 of their text (where $\frac{1}{2}$ should read $\frac{1}{c}$) and in several other terms in the description of the geometry and viscous damping expressions. Also, in deriving the nonlinear equations they present nonlinear kinematic equations which are actually incorrect because they neglect the rise and fall (pitch) of the bicycle due to variations in the steer angle. (The linearized versions of these equations are correct as shown in section 1 of our Appendix A, however, quadratic order terms are needed to derive the correct potential energy.) By eliminating the effects of viscous damping in the steering column, and making the above corrections N & F's final equations of motion can be brought into agreement with those derived in Chapter III.

N & F refer to Döhring[4] and state that their equations agree in form, but it is unlikely they meant term for term as we have found them to be in disagreement. They also refer to a Russian book by Loǐcjanskiǐ and Luré [1935] when analyzing a simplified model of an uncontrolled bicycle on p. 355. Because this reference was not available, it is not known if agreement actually exists, however it seems probable because N & F's equations become correct when simplified in this way. N & F do not mention any other bicycle-related works, although their massive reference list includes Carvallo [1901].

Singh and Goel, 1971

In January 1971 Singh and Goel (S & G) add steer damping to the Basic bicycle model in analyzing a Rajdoot motor scooter. In their analysis they claim to use Döhring's [1955] linearized equations of motion (which we have found to be correct) with a steering torque proportional to the the time derivative of the steer angle (viscous damping). We have not rigorously compared term by term but casual observation shows that the equations are in the same format as those of Döhring [1955].

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[4] See p. 361 of their text.

S & G refer to Pearsall [1922], Timoshenko [1948], Döhring [1955], Collins [1963], and Singh [1964], but make no comparison to their equations of motion.

Sharp, 1971

In August 1971 Sharp, who apparently began working on the equations of motion while at the B. S. A. motorcycle company, published a paper presenting his version of the linearized equations of motions for the motorcycle. In his Lagrangian approach rather than using the method presented by Neĭmark and Fufaev in Chapter III, he explicitly allows the vertical force from the ground on the front wheel (Z_f) to do work on the bicycle. For this reason Z_f appears in his expressions for the generalized forces. In this way he accounts for the change in potential energy of the bicycle when steered. The nonlinear equations he presents are actually only approximations for this reason.

Allowing for wheel side slip, and incorporating the work done by the vertical force on the front wheel, he derived Lagrange's equations with generalized forces at the wheels' contact with the ground. These resulted in four equations of motion, incorporating front and rear tire side forces, which govern lateral motion, yaw, roll, and steer of the motorcycle. They which appear in his paper starting at the bottom of p. 327 (no equation numbers are given). These equations are correct as far as we know.

However, when assuming that the tires have infinite stiffness (no side slip), which reduces the number of equations from four to two, an algebraic mistake and several typographical errors occur in the Appendix II. As a result the steer equation (the second equation) is incorrect. The algebraic error made by Sharp results in the incorrect cancellation of the following term (in his notation),

$$2[M_f ek + I_{fz}\cos\epsilon + M_f eb]l_1 t\ \ddot{\delta}$$

We also make note of the following typos: the x_1^2 in the lean equation of Appendix II should read $\dot{x}_1^2$; there is an extra parathesis in the ninth term of the fourth equation in Appendix 1 section entitled "Linear equations of motion"; the term $\dot{x}_1\cos\epsilon\ddot{\delta}$ in the expression for $\ddot{\psi}$ in Appendix II should read $\dot{x}_1\cos\epsilon\dot{\delta}$; I_{fy} should read i_{fy} in the $\dot{\phi}$ term of the steer equation of appendix 2; and finally terms involving $\frac{i_{fy}}{R_f}l_1 t\dot{x}_1\sin\epsilon$ in the $\dot{\delta}$ term of the steer equation can be eliminated as they cancel one another.

Sharp also makes the slightly restrictive assumption that one principal axis of the center of mass moment of inertia tensor of the front assembly is parallel to the steering axis. Thus the equations in his paper, when corrected, are a subset of those derived in Chapter III. Sharp refers to the work of Whipple [1899], Pearsall [1922] and Collins [1963], but does not compare his equations to theirs.

Roland, 1971

In 1971 Roland published a report written for the Schwinn Bicycle Company containing a extensive nonlinear computer simulation study.[5] In this report Roland derived nonlinear equations that represent the motion of a bicycle with tire side slip

[5] This report was based on work performed for a National Commission on Product Safety research contract. See Roland [1970].

and rider lean. His 8 equations of motion are shown in matrix form on p. 37 of his report. Reading from the top down the first three equations represent force balance for the entire bicycle. The fourth through the sixth equations represent moment balance for the entire bicycle. The seventh equation is apparently moment balance for the front assembly about the steer axis, which can presumably be used to solve for the steering torque if the tire side force is eliminated. The eighth and final equation represents the rider upper-body lean degree of freedom, and can be used to solve for the tilting moment of the bicycle on the rider when rider motion is prescribed. These equations are written so that the second time derivatives are all on the left side of the matrix equation, while all the lower order terms are on the right hand side.

Roland used axes parallel and perpendicular to the steering axis in the plane of the rear frame, and perpendicular to the rear frame. However, his report to the Schwinn bicycle company is missing an important figure describing the orientation of the body-fixed axes. This figure is contained in a later publication Mechanics and Sport [1973]. In the later publication, Roland also corrects some typos that were in the 1971 publication.

In the 1971 report, the seventh equation, the steer moment equation, is given on p. 13 as eq. (2.3.30). We took this equation and assumed $e = x''_{TF} - x''_F$ in order for it to agree with the seventh equation in the matrix on page 37. We then made simplifications to the equation to see if it agreed with our Lagrange equation for ψ, (3.10g).

First we set any term multiplied by y_F or y''_F equal to zero. (This means there is no lateral imbalance.) We then neglected terms multiplied by the pitch rates q and q'', which are second order effects. Next we assumed angles (and their time derivatives) to be small and let $\sin\delta = \delta$ and $\cos\delta = 1$, and cancelled any products of the small quantities p, p'', q, q'', and δ (and their time derivatives).

We then linearized the variables γ_{12}, γ_{22}, γ_{32} in the same way. Terms multiplied by γ_{12} become zero, $\gamma_{22} = m_f(rV - g\phi)$ and $\gamma_{32} = m_f g$.

The coefficient of the $\ddot{\delta}$ term seems correct, and the resulting equation appears somewhat similar to our equation (3.14), but we are not able to make the resulting equation agree completely. There is some question as to whether the comparison we are making is correct, because it is not understood if in fact Roland's equation (2.3.30) should be equivalent to our equation 3.10g.

An equation equivalent to our lean equation has not yet been constructed from Roland's equations. However, it is probable that an equivalent equation would be obtained by combining the fourth and sixth equations in his matrix to represent rolling moment about the track line, setting the rider lean angle to zero, including the mass of the rider with that of the rear frame and rear wheel, setting pitching motion to zero, and setting tire side slip to zero. The lateral forces, F_{ytr} and F_{ytf}, on the wheels can perhaps be solved for analytically using the first, fourth and sixth matrix equations or taken from other linearized equations studies. Since we have not attempted this, we are not able to judge whether his lean equation reduces to ours.

Roland refers to the works of Whipple [1899], Bower [1915], Pearsall [1922], Manning [1951], Döhring [1955], Collins [1963], Singh [1964], and also Singh and Goel [1971]. However, he makes no comparisons to their equations of motion.

Weir, 1972

In an appendix to his 1972 UCLA Ph.D. dissertation focusing mainly on the control and handling characteristics of motorcycles, Weir derived the equations of motion for the Basic bicycle model with a general Newtonian approach, linearizing as the derivation proceeded. Weir's final 4 equations, eq. [A-85], [A-92], [A-99], [A-108] in his analysis, represent the lateral motion, yaw, lean and steer equations of motion.

Weir was the only author to state explicitly that he compared his equations to another past work.[6] He compared his equations to Sharp's [1971] four equations (before Sharp's simplification to only two nontrivial degrees of freedom). In comparing Weir's 4 equations to Sharp's four equations, we find Weir and Sharp in agreement with one another. Weir, however, is more general than Sharp, in that he did not make the simplifying assumption regarding the principal axes of the front inertia. When Weir's four equation are simplified by adding the zero sideslip constraints we find his equations agree exactly with ours, as long as our nonstandard sign convention for wheel angular momentum (positive for forward rolling) is recognised.

[6] See page 130 of Weir's dissertation.

Besides stating that comparing his equations agree with to Sharp's, Weir refers to Whipple [1899], Pearsall [1922], Döhring [1955], Singh [1964], Singh and Goel [1971], but does not compare his equations to these works.

Singh and Goel, 1975

In 1975 Singh and Goel presented (but did not derive) a 12th order mathematical model, for the continued analysis of the Rajdoot scooter. Instead of using Singh's [1964] equations, or Döhring [1955] equations as they did in 1971, they employ a Lagrangian formulation which appears similar to Sharp's [1971] format. The authors claim that the model used is a fully general Basic bicycle model, having in addition unsymmetric lateral mass distribution, lateral slip, aerodynamic forces, viscous damping of the steering, and transient tire forces and moments (which account for the high order of the system).

The four equations of motion presented are said to represent the lateral motion, yaw, lean, and steer equations of motion. We have not yet checked these equations for correctness, but they do appear similar to Sharp's [1971] four equations. Singh and Goel refer to their 1971 paper on the Rajdoot scooter, and to Sharp's [1971] paper, but do not compare their equations.

Sharp and Jones, 1975

In 1975 Sharp and Jones use the equations derived by Sharp [1971] and modify it to incorporate a different tire model. As in the 1971 paper the principal axes of inertia are assumed to be parallel and perpendicular to the steering axis equations of motion. Other than this, these equations are equivalent to those in Sharp's 1971 paper, which when simplified correctly formed a subset of the equations presented in Chapter III of this thesis.

Weir and Zellner, 1978

Weir and Zellner later published the results of Weir's dissertation derivation in Motorcycle Dynamics and Rider Control (SP-428, 1978), but mistakenly thinking Weir's earlier derivation was wrong, they deleted a necessary term without comment. The term needing correction can be found on page 8 in the matrix equation (1), where the second row fourth column terms of the matrix should read,

$$\frac{L_{\ddot{\delta}}}{I_{xx}} s^2 + L_{\dot{\delta}} s + L_{\delta}$$

There are also some typos in equation (1) and we note the third row fourth column term should read,

$$N_{\ddot{\delta}} s^2 + N_{\dot{\delta}} s + N_{\delta}$$

and finally the fourth row fourth column term should read,

$$T_{\ddot{\delta}} s^2 + T_{\dot{\delta}} s + T_{\delta}$$

Because of these typographical errors we recommend using Weir's dissertation for any comparison of equations or results.

Incidentally, when corrected Weir and Zellner's matrix can be written to be symmetric except for the antisymmetric gyroscopic terms, but his notation does not make this evident.

Gobas, 1978

Using a technique which he calls the Boltzman Hamel method, in 1978 Gobas presented a linearized set of equations very similar in form to Neĭmark and Fufaev [1967]. Gobas' equations, (1.4) in his paper, incorporate the forward acceleration of the bicycle, $\dot{V}$. Setting $\dot{V}$ terms to zero and comparing, we think the lean equation may be correct, but in the steer equation the coefficient to the χ_r term seems to be in disagreement with the equations in our Chapter III. The variable b is not defined in the paper but we suspect that it is equivalent to our ν.

Gobas refers to Neĭmark and Fufaev, but does not compare equations.

Adiele, 1979

In his 1979 Master's thesis Adiele, who was focusing on design optimization and performance evaluation of two-wheeled vehicles, derived nonlinear equations of motion for the Basic bicycle with tire side slip using Kane's method of generalized active and inertia forces.

His equations, representing lateral motion, lean, steer, and yaw (in that order) are present in matrix form on pages 22-24 of his thesis. His variable V is our $\dot{X}_r$, λ is our χ_r, θ is our ψ, and r is our $\dot{\theta}_r$. Because his equations resembled Sharp's [1971] four equations, we expanded Adiele's matrix, linearized his equations for small values of λ and θ, and compared them to the equations in Sharp's [1971] Appendix I.

The results show that Adiele's equations are in error, missing several terms compared to Sharp and having several sign errors. However, by allowing the front mass to be zero his equations are nearly correct.

Adiele refers to Roland [1971], but does not compare equations. A subsequently published paper by Taylor and Adiele [1980] on stability in large angle steady turns also appears to rely on Adiele's equations, even though the authors evidently knew of earlier linearized studies (by Weir, and others) which they could have used to check their equations.

Lowell and Mckell, 1982

In 1982 Lowell and Mckell, using ad hoc arguments similar in style to Pearsall [1922] derive a set of linearized equations for a Basic bicycle model with a point mass in the rear part, some steering inertia and front gyroscopic effects, but no front mass, and no tilt of the steering axis. When compared to our equations simplified for this case, we find there is significant disagreement. Several terms have been neglected in both the lean and steer equation, however, the terms which are presented are correct. The neglected terms are significant, as a bicycle with vertical steering axis and positive trail *should* return upright if speed is great enough ($E > 0$), and show ever-increasing lean if speed is below a critical value ($E < 0$).[7] However their approximations make $E = 0$ always, so their bicycle model neither straightens up nor leans further, but in fact oscillates about a steady turn.

[7] For this simple bicycle E varies exactly opposite to E for a standard bicycle. When it is positive at low speeds and negative at high speeds.

We find the only way to make their equations correct is to use them for a bicycle with zero gyroscopic effects and zero trail.

Lowell and McKell refer to Timoshenko and Young [1948], Gray [1918], and Pearsall [1922] but only state (correctly) that their lean equation agrees with Timoshenko's when simplified. They made no other comparisons.

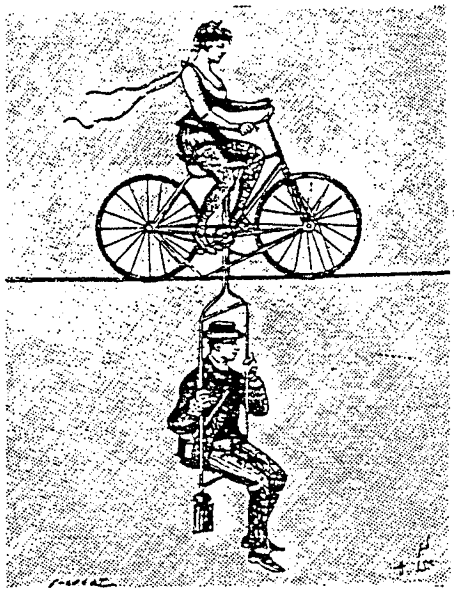
Conclusions

Of the 20 sets of equations discussed in this chapter only 3 sets (Döhring [1955], Singh and Goel's [1971] adaptation of these, and Weir [1972]) agreed exactly with those we presented in Chapter III of this thesis. (The slip angle condition had to be set to zero in Weir's equations.) Five others simply had minor errors, or were not as general (Whipple [1899], Carvallo [1901], Sommerfeld and Klein [1903], Timoshenko and Young [1948], and Sharp [1971]). Three (Collins [1963], Singh [1964], and Roland [1972]) were to difficult too evaluate, though we have definite reservations about the first two. The remaining eight were missing terms, or disagreed in other ways (we did not check Singh and Goel [1975]).

Other works which derived linearized equations of motion, but whose comparison results are not presented here, are Eaton [1973] and Psiaki [1979]. Eaton's derivation was not noticed until late in this thesis's progress. Psiaki derived very dense nonlinear equations and then linearized rather formally; we did not expend the effort to sort out his notation. Guo [1979] performed a nonlinear analyses but did not linearize, so we did not compare to his equations. Psiaki stated he found numerical agreement with Collins, and Guo referred to Neĭmark and Fufaev but made no comparison with them.

Other scientists have studied various aspects of bicycle behavior without deriving equations of motion. Rankine [1869] described steering phenomenology, and discussed the relation between a sinusoidal steering motion and the resulting sinsoidal lean angle. Sharp [1896] derived the steer torque in a steady turn.[8] Jones [1970] approximated the steer torque arising from bicycle lean ($K_{\chi\psi}$). Man and Kane [1979] studied steady turning at large lean angles.

[8] See eq. 6 on page 231 of his book. His result is nearly correct except that it neglects the effect of centrifugal force on the mass-center of the front assembly.



V. STABILITY ANALYSIS OF THE BASIC BICYCLE MODEL

The Meaning of Bicycle Stability

Traditionally when discussing dynamic stability of linear systems the concepts of equilibrium and degrees of freedom are used to define an equilibrium state of a system as being asymptotically stable or asymptotically unstable.[1] For the case of the bicycle we use these concepts to define stability in terms of only two of the bicycle degrees of freedom. For the stability of the bicycle we are only concerned with the stability of the variables representing the lean and the steer degrees of freedom, χ_r and ψ, respectively. That is, after the bicycle system is perturbed, θ_r and $\dot{X}_r$, can take on new nonzero equilibrium positions and the system, for all practical purposes, can still be considered stable.[2] Hence, when we discuss bicycle

[1] See the definition of stability on p. 3 of Etkin [1982].

[2] We lack mentioning the other generalized coordinates' time derivatives $\dot{Y}_r$, $\dot{\phi}_r$,

stability we apply the definition of dynamic stability only to the lean and steer (χ_r and ψ) degrees of freedom. We therefore define bicycle stability in the following way:

A bicycle is stable, if, after a very small disturbance from its vertical straight-ahead equilibrium motion it, asymptotically approaches a vertical straight-rolling configuration in the lean and steer degrees of freedom, χ_r and ψ, respectively.

This modified definition of dynamic stability, bicycle stability, assumes that for the degrees of freedom χ_r, and ψ, another equilibrium position does not exist close enough in the vicinity of the slightly disturbed region to cause the disturbed motion to converge to another, different, equilibrium position. From parking lot experiments this seems to be a good assumption for small disturbances.[3]

Before going on, we point out that just because a bicycle design configuration is found to be stable, does not necessarily imply that a rider would, or should, desire it more than an unstable bicycle design configuration. We emphasize that our definition of a self-stable bicycle configuration does not imply that a bicycle which is more comfortable, safer, and/or easy to ride. In fact, a self-stable bicycle configuration may seem overly sluggish and undesirable. The goal of this thesis is *not* try to gain understanding of whether a self-stable bicycle is easier to ride or

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and $\dot{\phi}_f$, which do not effect the linearized equations of the Basic bicycle model. Note however, these variables can also take on new equilibrium positions.

[3] Prior to the analytical studies done for this thesis many observational experiments were performed. Appendix D contains the announcement of a bicycle stability demonstration performed by the author highlighting the observations made.

more comfortable, but rather to understand analytically the effects of bicycle design parameters on mathematically based stability criteria. Further studies, beyond the scope of the thesis, are required to compare rider impressions to our results.

Stability Analysis Techniques

Historically, stability of the vertical straight ahead (upright) equilibrium configuration of the bicycle and motorcycle has been studied in four ways: analysis of the eigenvalues and eigenvectors of the system; numerical integration of the equations of motion and study of the solutions; application of the Routh-Hurwitz criteria; or experimental observations of bicycle behaviour. Each approach has its merits and drawbacks, as we will now explain.

First, using the system of equations, or the characteristic fourth order polynomial derived in Chapter III of this thesis, one can determine the eigenvalues (the roots to the fourth order equation), and eigenvectors (mode shapes) of the system which can be used to calculate the natural frequencies and mode shapes of the system representing a bicycle traveling at a particular speed.[4] If, in such an analysis, any of the eigenvalues have positive real parts, the solution to the system will grow away from the equilibrium in time and the system is unstable (at least based on the linearized equations). Complex eigenvalues represent oscillatory solutions whose real parts determine whether the amplitude of the oscillations will grow or decay

[4] When we say system, we are referring to the set of equations that describe the bicycle's motion.

in time.[5] Thus eigenvalue-eigenvector analyses allow for various design configurations to be compared by numerically evaluating their eigenvalues. From this we can determine which design configuration is mathematically more or less stable.[6] Traditionally, this has been done by saying the more negative the real part of the eigenvalue, the more stable the system. The eigenvectors can be used to try to imagine how the bicycle went unstable. That is, we can try to distinguish how each degree of freedom is contributing to the instability, falling over and steering back and forth. It is from the eigenvalue-eigenvector stability analyses that the terms capsize, weave and wobble modes have been adopted to described the motion of the bicycle or motorcycle at various speeds.[7] Eigenvalue-eigenvector analyses can only be performed with a linearized set of equations.

A second method used to study stability is to numerically solve the equations of motion using a digital or analog computer. This can be done for linear or nonlinear sets of equations. Plots of the response (the solution) to various inputs can be used to quantitatively and qualitatively characterize the stability of the system, and/or changes in design parameters (just as with the eigenvalue-eigenvector analysis.).

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[5] For any polynomial equation with real coefficients, complex roots will always occur in complex conjugate pairs.

[6] Calculating the eigenvalues also allows the half-life of the amplitude of the oscillation and damping ratio to be calculated, which can be used as a measure of stability for a particular design configuration, instead of the eigenvalue.

[7] See Sharp [1975]. Note that the wobble mode is only present when a tire model allowing for tire slip angle is incorporated into the equations.

This method can also be used to verify stable equilibrium motion(s), if any exist.

One of the major problems associated with numerical studies is the verification of the equations themselves. Because nonlinear equations are generally more complicated to solve for than linear equations, we believe the probability of mathematical error is higher with nonlinear models (and quite possibly little further understanding is gained). Thus, although the computer may be powerful enough to solve nonlinear equations, the results should be reviewed with caution until the nonlinear equations have been verified in some way.

Another computer-related method for studying stability of the bicycle would be modeling the bicycle components as rigid bodies and constraining motion between the rigid bodies by springs with varying stiffness. The computer could then be used to solve the equations of motion of individual rigid bodies, without complicated dynamical equations. To the author's knowledge no one has performed this type of numerical simulation for bicycles.

A third method of evaluating bicycle stability is to apply the Routh-Hurwitz criteria, which is the method used in this thesis. This method allows for the stability of an equilibrium configuration to be determined based on the *coefficients* of the fourth order polynomial derived in Chapter III. It determines whether any of the eigenvalues (roots to the fourth order polynomial) have real positive parts, *without* actually solving for them. This technique yields criteria which directly lead to analytical expressions linked to the stability of the system. Qualitative statements can be made by comparing various design configurations stability regions. Quantitative statements can be formulated from analytical expressions linked to stability. Like eigenvalue and eigenvector analyses, the Routh Hurwitz criteria are applicable only to the linearized system of equations, as it is based on the fourth order polynomial, which is the characteristic equation for the two coupled linearized differential equations.

The Routh-Hurwitz criteria are limited in the quantitative aspect, in that, it does not give any exact measure of *how* stable a *particular* design configuration is relative to another. However, as will be seen, it develops analytical expressions which give the relative importance of various design parameters for stability. In this way, quantitative insight into the important parameters in design is seen and provides an analytic foundation for further analysis of the parameters related to stability.

The fourth method used to analyze the stability of the bicycle is to experimentally measure the bicycle's behaviour. Data for various design configurations can then be compared and results analyzed. This thesis does not discuss the experimental methods used in analyzing bicycle stability or their results. The interested reader is referred to the work of Kondo [1955], Kageyama [1962], Kondo [1962], Fu [1966], Roland [1970], Jones [1970], and Eaton [1973].

Discussion of Analysis Techniques

In both the eigenvalue-eigenvector and numerical stability analyses, no immediate insight to the effect of design changes on the stability of the bicycle is likely to be apparent until after numerous bicycle configurations have been compared. Only after comparing trends in changes of the eigenvalues, or solutions to the equations can results be quantified to try to develop design criteria for achieving stability. This process has the potential to become long and tedious, and is subject to interpretation of data. Certainly, no definite proofs can be made from these techniques, and parameters of importance can not be seen without numerous computer simulations and study of results.[8]

Even with a detailed parameter study using the method of root loci, or nonlinear solutions to the equations of motion, results drawn from these kind of methods have yielded mostly numerical results for a particular design of bicycle or motorcycle, and do not provide the analytic design criteria needed by bicycle designers. It is the opinion of the author that studying more detailed models is not valuable until basic, analytically based, quantitative understanding of the importance of design parameters is available. Because of simpler computations and its direct link to analytically based design criteria as functions of velocity, application of the Routh Hurwitz stability tests were chosen as the main technique used in this thesis.

We next derive the Routh-Hurwitz criteria for the Basic bicycle model.

The Routh-Hurwitz Stability Criteria

The Routh-Hurwitz method differs from eigenvalue-eigenvector analysis because it tells us whether or not there are unstable eigenvalues, i.e. roots to a characteristic equation with positive real parts, without actually solving for them.

[8] Although not summarized in this thesis we reviewed others results and found few gave general conclusions about design for stability.

Results are analytical, hence useful for general statements and insight into important parameters in the design of bicycles. However, results do not give the same information on the degree of stability or instability provided by eigenvalues and eigenvectors.

The Routh-Hurwitz criteria are formulated using the coefficients to the fourth order polynomial (eq. 3.17). When we apply the criteria to the Basic bicycle model's characteristic equation the resulting necessary but not sufficient conditions are that the coefficients A-E all must be greater than zero. These conditions must be met for the system to be stable.[9] Using the notation introduced in Chapter III, these conditions are the first five of seven Routh-Hurwitz criteria that must be satisfied for the system to be stable. In general they are,

$$A = \frac{1}{2}\,\mathcal{M} * \mathcal{M} = det\ \mathcal{M} > 0 \tag{4.1}$$

$$B = \mathcal{M} * \mathcal{C} > 0 \tag{4.2}$$

$$C = \mathcal{M} * \mathcal{K} + det\ \mathcal{C} > 0 \tag{4.3}$$

$$D = \mathcal{K} * \mathcal{C} > 0 \tag{4.4}$$

$$E = \frac{1}{2}\,\mathcal{K} * \mathcal{K} = det\ \mathcal{K} > 0 \tag{4.5}$$

where the matrix multiplication $*$ is defined in Chapter III and the quantities $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{K}$ are the mass, damping , and stiffness matrices for the system as follows,

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{\chi\chi} & \mathcal{M}_{\chi\psi} \\ \mathcal{M}_{\psi\chi} & \mathcal{M}_{\psi\psi} \end{pmatrix} = \begin{pmatrix} T_{yy} & F'_{\lambda y} + \frac{c_f}{c_w} T_{yz} \\ F'_{\lambda y} + \frac{c_f}{c_w} T_{yz} & F'_{\lambda\lambda} + 2\frac{c_f}{c_w} F''_{\lambda z} + \frac{c_f^2}{c_w^2} T_{zz} \end{pmatrix}$$

[9] See Ogata [1976].

$$\mathcal{C}=\begin{pmatrix} \mathcal{C}_{\chi\chi} & \mathcal{C}_{\chi\psi} \\ \mathcal{C}_{\psi\chi} & \mathcal{C}_{\psi\psi} \end{pmatrix}=$$

$$\begin{pmatrix} 0 & -\left(H_f\cos\lambda+\frac{c_f}{c_w}H_t\right)+V\left(T_{yz}\frac{\cos\lambda}{c_w}-\frac{c_f}{c_w}m_t\overline{h}_t\right) \\ H_f\cos\lambda+\frac{c_f}{c_w}H_t & V\left(\frac{\cos\lambda}{c_w}\left(F''_{\lambda z}+\frac{c_f}{c_w}T_{zz}\right)+\frac{c_f}{c_w}\nu\right) \end{pmatrix}$$

$$\mathcal{K}=\begin{pmatrix} \mathcal{K}_{\chi\chi} & \mathcal{K}_{\chi\psi} \\ \mathcal{K}_{\psi\chi} & \mathcal{K}_{\psi\psi} \end{pmatrix}=$$

$$\begin{pmatrix} -gm_t\overline{h}_t & g\nu-H_tV\frac{\cos\lambda}{c_w}-V^2\frac{\cos\lambda}{c_w}m_t\overline{h}_t \\ g\nu & -g\nu\sin\lambda+VH_f\sin\lambda\frac{\cos\lambda}{c_w}+V^2\frac{\cos\lambda}{c_w}\nu \end{pmatrix}$$

As was shown in Chapter III of this thesis, and shown above, in general, the coefficients A-E are functions of the coefficients of the lean and steer equations (3.14,15), which are functions of the velocity of the bicycle. (This proves the intuitive result that the stability of the bicycle is dependent on the speed at which it is moving.) In what follows, for a given bicycle design we will imagine evaluating the stability criteria (4.1-4.5) at many different velocities, to find 'stable regions.'[10]

In addition to the coefficients A-E being positive, in order to have a stable configuration (a set of design parameters, which has a range of stable velocities) two additional criteria must be met. These criteria are functions of the coefficients

[10] When we say region of stability we mean that for a range of velocities, all criteria are satisfied. In general, some lower and higher speeds will bound the stability region.

A-E and once derived are as follows,[11]

$$BC - AD > 0 \tag{4.6}$$

$$D > \frac{B^2E}{BC - AD} \tag{4.7}$$

If we further assume that a region of stability exists (A, B, C, D, E, 4.6, and 4.7 are all met for a range of velocities), we find equation (4.6) is encompassed in equation (4.7). That is, while in a region of stability, (4.7) is a more restrictive criterion than (4.6), and is always met if (4.7) is met. Thus, assuming a region of stability exists, equation (4.6) is always met and equation (4.7) can be rewritten as,[12]

$$BCD > AD^2 + EB^2 \tag{4.7a}$$

or equivalently,

$$C > A\left(\frac{D}{B}\right) + E\left(\frac{B}{D}\right) \tag{4.7b}$$

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[11] For brevity, we have skipped the derivation of these criteria but rules for their derivation can be found in Ogata [1976].

[12] Based on the results of previous studies and experiments it is reasonable to assume that some region of stability exists for the Basic bicycle model with some design configurations. In this study we will always assume that a region of stability exists when discussing the stability criterion (4.6) and (4.7), then verify our assumption numerically by evaluating (4.1-7).

Interpretation and Meaning of the Routh-Hurwitz Criteria

For a given design configuration which has a range of stable velocities, there must be upper and lower limiting speeds which bound the stable regions.[13] For a general design, it is conceivable that more than one stable region can exist, as will be explained. This section describes which Routh-Hurwitz criteria are the limiting criteria for these stable regions. That is, for a given design configuration, if in a range of stable velocities, which criteria will first be violated when leaving the stable region.

The Routh-Hurwitz stability criteria are essentially a way of indicating when the eigenvalues of a system (roots to the fourth order characteristic equation) have negative real parts. To understand which Routh-Hurwitz criteria are the limiting criteria, we studied when the roots of the fourth order polynomial goes from describing a stable system (all roots having zero or negative real parts) to an unstable system (any roots having positive real parts).

Graphing the roots to the characteristic equation in general, as shown in figure 5.1, we note that a stable system can become unstable in two ways:

1) a negative purely real root becomes positive or,

2) the negative real parts of complex conjugate roots become positive.

---

[13] The upper and lower stability limit could conceivably be infinity or zero, respectively. As we will mention later, from a physical standpoint we know that traditional bicycle design configurations are unstable with zero velocity. One might investigate the stability criteria to see if adding gyros, etc. to the Basic bicycle model could make it self-stable with zero velocity.

Point P_1 illustrates that when a purely real root changes from negative to positive, it must pass through zero. We note from the form of the characteristic equation (3.17) that the *only* time that a purely real root can equal zero is when the E coefficient is zero. (Substitute $s = 0$ into the equation (3.17).) This means that assuming some region of stability exists, the E coefficient, equation (4.5), can be a limiting criterion. In this case, for example, as velocity changes the E criterion will be the first violated when entering an unstable velocity region (with either velocity increasing or decreasing).

Points P_2 and P_3 illustrate that when a conjugate pair of complex roots change from negative real parts to positive real parts, they must pass through the purely imaginary condition. Substituting the component of a purely imaginary theoretical root, say $i\omega$, into the fourth order polynomial we find that the condition that such a solution exist yields the equality of inequality (4.7a), as is next shown.

Substituting $i\omega$ for s in equation (3.17) we have,

$$A\omega^4 - iB\omega^3 - C\omega^2 + iD\omega + E = 0$$

Equating real and imaginary components to zero,

$$A\omega^4 - C\omega^2 + E = 0 \tag{4.8}$$

$$B\omega^3 - D\omega = 0 \tag{4.9}$$

Solving for ω^2 in (4.9) and substituting this into (4.8) we have, the stability limit for Routh-Hurwitz criterion (4.7a),

$$CDB = AD^2 + EB^2$$

This shows mathematically that if a stability region exists, (4.7a) also can be a limiting criterion. The E inequality (4.5), and the Routh-Hurwitz inequality (4.7a) together will detect all transitions to instability, and in fact it can be shown that one of these inequalities must be violated *before* any other criteria can be. Thus, one or both determine the limits to any stable range of velocities.

To determine whether (4.5) or (4.7a) could ever provide both of the limiting criteria for a stable region, we look at the general form of (4.5) and (4.7a) as functions of velocity. Expanding the E inequality (4.5), in terms of the bicycle parameters as a function of velocity, we find a function of V of the form,

$$E = e_0 + e_2 V^2$$

where e_0 and e_2 are constants based on the design parameters of the bicycle. Because E has no term in V to first order, we know from the shape of the curve that E can only provide *one* of the limiting velocities, not both. That is, it can only have, at most, one positive velocity at which it changes sign as is indicated in figure 5.2. So E might serve as a lower limit or upper limit to some stability region, but not both. If e_0 and e_2 are both positive, equation (4.5) is not a limiting criteria. Meaning, should a region of stability exist, (4.7a) must be both the upper and lower stability limits as is next explained.

Note that in inequality (4.7a), that if we set either B, C, or D (inequalities 4.1-4) to zero, that in each case inequality (4.7a) is violated. This means that provided a region of stability exists and *the coefficient A is positive*, that (4.7a) must be one or both of the limiting criteria because it will be violated before (4.1-4). Expanding the general form of (4.7a) in terms of the bicycle parameters we can see that this is possible. We find (4.7a) of the form,

$$V^2(r_4V^4 + r_2V^2 + r_0) > 0$$

where r_4, r_2, and r_0 are constants based on the design parameters of the bicycle. As shown in figure 5.3, in general equation (4.7a) criterion can define zero, one or two stability regions by itself due to the form of its dependence on velocity. Should (4.7a) only provide one limiting criteria the E criteria can act as the other limiting criteria if a stability region exists. Figure 5.4 summarizes the various possible combinations between criteria (4.5) and (4.7a) for determining a stability region in the positive velocity region.

Reviewing figure 5.4 and assuming that the bicycle is unstable at zero velocity we have four possible combinations of the E criterion and the criterion (4.7a) as a function of velocity:

1) always Unstable (U)

2) Unstable, then always Stable (US)

3) Unstable, then Stable, then always Unstable (USU)

4) Unstable, then Stable, then Unstable, then always Stable (USUS)

Of course some of the combinations presented in figure 5.4 may be physically impossible, but that remains to be proven.

As just mentioned, for (4.7a) to absolutely be a limiting criterion the coefficient A must be positive. Before going on we note (and show later) that A, the determinant of the mass matrix, is a constant and not a function of velocity. If then A is positive, it is positive for all speeds and the condition $A > 0$ is always met. We assume then for analysis purposes that A is always positive.[14]

Equations $(4.1 - 7a)$ form the analytical basis for stable bicycle design criteria. We will next expand each of the necessary conditions $(4.1 - 5)$ out in terms of the bicycle design parameters to formulate and simplify analytic design criteria for the bicycle. We found criterion (4.7a) was very complicated and have only represented this as functions of the velocity coefficients found in inequalities (4.1-5). After developing the design criteria some common beliefs about bicycle stability will be disproved and the relative importance of some design parameters will be illustrated. Later in Chapter VI, we suggest how these stability criteria can be used in design practice to optimize the stability of the bicycle.

Routh-Hurwitz Criteria for the Basic Bicycle Model

In order to aid our analysis the equations (4.1-4.5) have been expanded, first leaving the components of the $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{K}$ matrices intact and then showing simplified forms of the coefficients in terms of the bicycle design parameters. Note how the two constants related to the steering axis tilt and mechanical trail, $\frac{\cos\lambda}{c_w}$ and $\frac{c_f}{c_w}$, repeatedly occur in these expressions. These constants naturally come out of the expansion of the criteria and also frequently appear in the equations of motion of the bicycle.

[14] The author has been assured by Dr. Jim Papadopoulos that there is an analytical proof showing the determinant of the mass matrix always must be positive. The author did not have time to reference this proof.

The reader should note that the simplified forms of the coefficients are separated according to their coefficients $\frac{c_f}{c_w}$, $\frac{\cos\lambda}{c_w}$, $\frac{\cos\lambda}{c_w}\frac{c_f}{c_w}$, and $\left(\frac{c_f}{c_w}\right)^2$. In some cases this is a measure of the relative size of the particular term compared to other terms in the equations. In general, however, this is not always the case. The following are the results of these simplifications,

$$A = det\ \mathcal{M} = a_0 > 0 \tag{4.1}$$

$$= T_{yy}\left(F'_{\lambda\lambda} + 2\frac{c_f}{c_w}F''_{\lambda z} + \frac{c_f^2}{c_w^2}T_{zz}\right) - (F'_{\lambda y} + \frac{c_f}{c_w}T_{yz})^2 > 0 \tag{4.1a}$$

$$= T_{yy}F'_{\lambda\lambda} - {F'_{\lambda y}}^2 + 2\frac{c_f}{c_w}\left(T_{yy}F''_{\lambda z} - F'_{\lambda y}T_{yz}\right) + \left(\frac{cf}{c_w}\right)^2 det\,\mathbf{T} > 0 \tag{4.1b}$$

$$B = \mathcal{M} * \mathcal{C} = b_1 V > 0 \tag{4.2}$$

$$T_{yy}V\left(\frac{\cos\lambda}{c_w}\left(F''_{\lambda z} + \frac{c_f}{c_w}T_{zz}\right) + \frac{c_f}{c_w}\nu\right)$$

$$-\left(F'_{\lambda y} + \frac{c_f}{c_w}T_{yz}\right)\left(H_f\cos\lambda + \frac{c_f}{c_w}H_t\right)$$

$$-\left(F'_{\lambda y} + \frac{c_f}{c_w}T_{yz}\right)\left(-\left(H_f\cos\lambda + \frac{c_f}{c_w}H_t\right) + V\left(T_{yz}\frac{\cos\lambda}{c_w} - \frac{c_f}{c_w}m_t\overline{h}_t\right)\right)$$

$$> 0 \tag{4.2a}$$

$$= T_{yy}V\frac{\cos\lambda}{c_w}F''_{\lambda z} + T_{yy}V\frac{\cos\lambda}{c_w}\frac{c_f}{c_w}T_{zz} + T_{yy}V\frac{c_f}{c_w}\nu - T_{yz}V\frac{\cos\lambda}{c_w}F'_{\lambda y}$$

$$+ F'_{\lambda y}V\frac{c_f}{c_w}m_t\overline{h}_t - T_{yz}^2V\frac{c_f}{c_w}\frac{\cos\lambda}{c_w} + T_{yz}m_t\overline{h}_t\frac{c_f}{c_w}^2V > 0 \tag{4.2b}$$

$$= V\left(\frac{\cos\lambda}{c_w}\left(T_{yy}F''_{\lambda z} - T_{yz}F'_{\lambda y}\right) + \frac{\cos\lambda}{c_w}\frac{c_f}{c_w}\,det\,\mathrm{T}\right.$$

$$\left. + \frac{c_f}{c_w}\left(T_{yy}m_fd + F'_{\lambda y}m_t\overline{h}_t\right) + \left(\frac{c_f}{c_w}\right)^2\left(T_{yy}m_t\overline{l}_t + T_{yz}m_t\overline{h}_t\right)\right) > 0 \tag{4.2c}$$

$$
\begin{aligned}
C = \mathcal{M} * \mathcal{K} + det\, C &= c_0 + c_2 V^2 && (4.3)\\
= T_{yy} &\left(-g\nu \sin\lambda + V H_f \sin\lambda \frac{\cos\lambda}{c_w} + V^2 \frac{\cos\lambda}{c_w}\nu\right) \\
&- \left(F'_{\lambda y} + \frac{c_f}{c_w} T_{yz}\right) g\nu \\
&- \left(F'_{\lambda y} + \frac{c_f}{c_w} T_{yz}\right)\left(g\nu - H_t V \frac{\cos\lambda}{c_w} - V^2 \frac{\cos\lambda}{c_w} m_t \overline{h}_t\right) \\
&+ \left(F'_{\lambda\lambda} + 2\frac{c_f}{c_w} F''_{\lambda z} + \frac{c_f^2}{c_w^2} T_{zz}\right)\left(-g m_t \overline{h}_t\right) \\
&- \left(H_f \cos\lambda + \frac{c_f}{c_w} H_t\right)\left(-\left(H_f \cos\lambda + \frac{c_f}{c_w} H_t\right) + V\left(T_{yz}\frac{\cos\lambda}{c_w} - \frac{c_f}{c_w} m_t \overline{h}_t\right)\right) \\
&> 0 && (4.3a)\\
= V^2 &\left[\frac{H_f^2}{V^2}\cos^2\lambda\right. \\
&+ \frac{\cos\lambda}{c_w}\left(m_f d T_{yy} + \frac{H_t}{V} F'_{\lambda y} + F'_{\lambda y} m_t \overline{h}_t - \frac{H_f}{V} T_{yz}\cos\lambda + \frac{H_f}{V}\sin\lambda T_{yy}\right) \\
&+ \frac{c_f}{c_w}\left(\frac{2 H_f H_t}{V^2}\cos\lambda + \frac{H_f}{V} m_t \overline{h}_t \cos\lambda\right) \\
&+ \frac{\cos\lambda}{c_w}\frac{c_f}{c_w}\left(T_{yy} m_t \overline{l}_t + m_t \overline{h}_t T_{yz}\right) \\
&\left.+ \left(\frac{c_f}{c_w}\right)^2\left(\frac{H_t}{V} m_t \overline{h}_t + \frac{H_t^2}{V^2}\right)\right] \\
&+ \left[-F''_{\lambda\lambda} g m_t \overline{h}_t - 2F'_{\lambda y} g m_f d - T_{yy} g \sin\lambda m_f d\right. \\
&+ \frac{c_f}{c_w}\left(-2F'_{\lambda y} g m_t \overline{l}_t - 2T_{yz} g m_f d - 2F''_{\lambda z} m_t \overline{h}_t g\right) \\
&+ \frac{c_f}{c_w}\sin\lambda\left(-T_{yy} g m_t \overline{l}_t\right) \\
&\left.+ \left(\frac{c_f}{c_w}\right)^2\left(-T_{zz} g m_t \overline{h}_t - 2T_{yz} g m_t \overline{l}_t\right)\right] \\
&> 0 && (4.3b)
\end{aligned}
$$

$$D = \mathcal{C} * \mathcal{K} = d_1 V + d_3 V^3 \tag{4.4}$$

$$= -g\nu\left(-\left(H_f \cos\lambda + \frac{c_f}{c_w} H_t\right) + V\left(T_{yz}\frac{\cos\lambda}{c_w} - \frac{c_f}{c_w} m_t \overline{h}_t\right)\right)$$

$$- \left(g\nu - H_t V \frac{\cos\lambda}{c_w} - V^2 \frac{\cos\lambda}{c_w} m_t \overline{h}_t\right)\left(H_f \cos\lambda + \frac{c_f}{c_w} H_t\right)$$

$$- g m_t \overline{h}_t V\left(\frac{\cos\lambda}{c_w}\left(F''_{\lambda z} + \frac{c_f}{c_w} T_{zz}\right) + \frac{c_f}{c_w}\nu\right) \tag{4.4a}$$

$$= V^3\left[\frac{H_f}{V}\frac{\cos\lambda}{c_w} m_t \overline{h}_t \cos\lambda + \frac{\cos\lambda}{c_w}\cos\lambda \frac{H_f H_t}{V^2} + \frac{\cos\lambda}{c_w}\frac{c_f}{c_w}\left(\frac{H_t^2}{V^2} + \frac{H_t}{V} m_t \overline{h}_t\right)\right]$$

$$+ V\left[\frac{\cos\lambda}{c_w}\left(-g m_t \overline{h}_t F''_{\lambda z} - g T_{yz} m_f d\right)\right.$$

$$\left. + \frac{\cos\lambda}{c_w}\frac{c_f}{c_w}\left(-g m_t \overline{h}_t T_{zz} - g T_{yz} m_t \overline{l}_t\right)\right] > 0 \tag{4.4b}$$

$$E = \mathcal{K} * \mathcal{K} = e_0 + e_2 V^2 \tag{4.5}$$

$$= - g m_t \overline{h}_t \left(-g\nu \sin\lambda + V H_f \sin\lambda \frac{\cos\lambda}{c_w} + V^2 \frac{\cos\lambda}{c_w} \nu \right)$$

$$- g\nu \left(g\nu - H_t V \frac{\cos\lambda}{c_w} - V^2 \frac{\cos\lambda}{c_w} m_t \overline{h}_t \right) > 0 \tag{4.5a}$$

$$= \frac{V g \cos\lambda}{c_w} \left(H_t \nu - m_t \overline{h}_t H_f \sin\lambda \right) + g^2 \nu \left(m_t \overline{h}_t \sin\lambda - \nu \right) > 0 \tag{4.5b}$$

$$= V^2 g \left[\frac{\cos\lambda}{c_w} \left(-m_t \overline{h}_t \frac{H_f}{V} \sin\lambda + \frac{H_t}{V} m_f d \right) + \frac{\cos\lambda}{c_w} \frac{c_f}{c_w} \left(\frac{H_t}{V} m_t \overline{l}_t \right) \right]$$

$$+ g^2 \left[m_t \overline{h}_t \sin\lambda m_f d - m_f^2 d^2 \right.$$

$$+ \frac{c_f}{c_w} \left(m_t^2 \overline{h}_t \overline{l}_t \sin\lambda - 2 m_f d m_t \overline{l}_t \right)$$

$$\left. - \left(\frac{c_f}{c_w} \right)^2 \left(m_t^2 \overline{l}_t^2 \right) \right] > 0 \tag{4.5c}$$

$$(4.7a) = V^2(r_4 V^4 + r_2 V^2 + r_0) > 0 \tag{4.7a}$$

where,

$$r_4 = b_1 c_2 d_3 - a_0 d_3^2 \tag{4.8}$$

$$r_2 = b_1 c_0 d_3 + b_1 c_2 d_1 - 2 a_0 d_1 d_3 - e_2 b_1^2 \tag{4.9}$$

$$r_0 = b_1 c_0 - a_0 d_1 \tag{4.10}$$

and a_0, b_1, c_0, c_2, d_1, d_3, e_0, and e_2 are defined in equations (4.1–5). They represent the coefficients of the velocity when criteria A–E expanded as functions of velocity. The subscript representing the power of velocity.

Design Criteria and Examples for Simplified Cases

Equations (4.1-5) must be met for self-stability of a Basic bicycle model. In addition (4.7a) must be met to fulfill all Routh-Hurwitz criteria and guarantee self-stability. These equations, although complicated, are design criteria for stability of the bicycle in terms of bicycle design parameters. If all criteria (4.1-4.5) are met, and in addition (4.7a) is met, the Basic bicycle model is stable for small disturbances.

In order to gain understanding about bicycle stability, we desire to know the effects of the bicycle design parameters on the Routh-Hurwitz criteria. That is, what changes increase or decrease bicycle self-stability. Using the Routh-Hurwitz criteria stability can be measured three ways:

1) The size of the stable velocity region.

2) The minimal speed stability is achieved.

3) Analytically, by studying the relations between parameters in the criteria.

For our analysis we have chosen a primitive bicycle lacking several of what in the past have been deemed as making the bicycle stable. Our primitive model is a subset of the Basic bicycle model and, because it lacks several design features, is easier to analytically study. The primitive bicycle has no steering axis tilt, $\lambda = 0$, no mechanical trail, $c_f = 0$, no angular momentum, $H_t = H_f = 0$, and the front center of mass is located on the steering axis, $d = 0$. The inertias, however, are fully general. The primitive bicycle with tiny skates as wheels is shown in figure 5.5.

Next we will see how stable the primitive bicycle is by evaluating the stability criteria (4.1-5).[15] For the primitive bicycle model equations (4.1-5) simplify to:

$$A = T_{yy}F'_{\lambda\lambda} - {F'_{\lambda y}}^2 > 0$$

$$= T_{yy}F_{\overline{zz}} - F^2_{\overline{yz}} > 0$$

$$= \left(m_r\overline{h}_r^2 + R_{\overline{yy}} + m_f\overline{h}_f^2\right)F_{\overline{zz}} + det\ \mathbf{F} > 0$$

$$B = V\frac{1}{c_w}(T_{yy}F''_{\lambda z} - T_{yz}F'_{\lambda y}) > 0$$

$$= \frac{V}{c_w}\Bigg((R_{\overline{yy}} + m_r\overline{h}_r^2 + m_f\overline{h}_f^2)F_{\overline{zz}}$$

$$+ (-R_{\overline{yz}} + m_r\overline{h}_r\overline{l}_r + m_f\overline{h}_f(\overline{l}_f + c_w))F_{\overline{yz}}\Bigg) + det\ \mathbf{F} > 0$$

$$C = V^2\frac{1}{c_w}F_{\overline{yz}}m_t\overline{h}_t - F_{\overline{zz}}gm_t\overline{h}_t > 0$$

$$D = -V\frac{1}{c_w}gm_t\overline{h}_tF_{\overline{zz}} > 0$$

$$E = 0$$

15 Analytically we felt (4.7a) was too complicated to present. Instead we study (4.1-5) and then numerically check if and when criteria (4.7) is met.

For the primitive bicycle we see that we can prove that A is always positive. B is always positive if $F_{yz} > 0$ and $R_{yz} < 0$.[16] We note that F_{yz} must be positive, for C to be positive. D is always violated as it is always negative or zero. And finally, E is always violated as it is always equal to zero, but never negative.

Because the D and E criterion are *always* violated the primitive bicycle will *never* be self-stable.[17]

How then can this primitive bicycle be made stable?

We next add wheels to our primitive model. This resembles a real bicycle much more and can be thought of as a prototype machine built by someone who has never seen a modern bicycle. It is shown in figure 5.6. For the primitive model with wheels we find that equations (4.1-5) take the form:

[16] However, B could still be positive if $F_{yz} < 0$ and/or $R_{yz} > 0$, its just not guaranteed.

[17] In the remainder of this Chapter when we say self-stable we are referring to the Basic bicycle model model's equations being asymtotically stable for the lean and steer degrees of freedom.

$$A = T_{yy}F'_{\lambda\lambda} - {F'_{\lambda y}}^2 > 0$$

$$= T_{yy}F_{\overline{zz}} - F^2_{\overline{yz}} > 0$$

$$= \left(m_r\overline{h}_r^2 + R_{\overline{yy}} + m_f\overline{h}_f^2\right)F_{\overline{zz}} + det\ \mathbf{F} > 0$$

$$B = V\frac{1}{c_w}(T_{yy}F''_{\lambda z} - T_{yz}F'_{\lambda y}) > 0$$

$$= \frac{V}{c_w}\Bigg((R_{\overline{yy}} + m_r\overline{h}_r^2 + m_f\overline{h}_f^2)F_{\overline{zz}}$$

$$+ (-R_{\overline{yz}} + m_r\overline{h}_r\overline{l}_r + m_f\overline{h}_f(\overline{l}_f + c_w))F_{\overline{yz}}\Bigg) + det\ \mathbf{F} > 0$$

$$C = V^2(\frac{H_f^2}{V^2} + \frac{1}{c_w}\frac{H_t}{V}F_{\overline{yz}} + F_{\overline{yz}}m_t\overline{h}_t - \frac{H_f}{V}T_{yz})) - F_{\overline{zz}}gm_t\overline{h}_t > 0$$

$$D = V^3\left(\frac{1}{c_w}\frac{H_f}{V}m_t\overline{h}_t + \frac{1}{c_w}\frac{H_fH_t}{V^2} - V\frac{1}{c_w}gm_t\overline{h}_tF_{\overline{zz}}\right) > 0$$

$$E = 0$$

Note that A and B are not functions of the angular momentum of the wheels due to spinning. Hence, they do not change from the primitive case. The criteria C will definitely be positive as speed increases because the coefficient to the V^2 term is always positive.[18] The D criteria will definitely become positive at some velocity for the same reasoning. E, however, is still equal to zero (almost positive) so the primitive bicycle with positive momentum gyroscopes (or wheels), will also *never* be self-stable. We next analyze the E, equation (4.5*b*), criteria to see what is the slightest change that can be made to the primitive bicycle with wheels so that it becomes self-stable.

We note that the slightest amount of negative steering axis tilt, $-\lambda$, will make this criterion positive. And although this complicates the other criteria (4.1-4,7) we have programmed equations (4.1-5,7) to show that this modified primitive bicycle model is *almost always* self-stable.[19] Figure 5.8 shows the primitive bicycle with wheels and a slight negative steering axis tilt. This bicycle showed it was self-stable from 6.0 to 50+ mph for inertia properties approximating a 200 lb. rider as shown in figure 5.7.

In addition, we found for the given inertia configuration, that the primitive bicycle with wheels could be made stable by *individually* adding mechanical trail,

---

[18] This assumes we are talking about a normal bicycle with no "reverse spinning" gyroscopes attached to make the magnitude of angular momentum of the wheels negative using our notation.

[19] For the assumed inertia distribution shown in figure 5.7.

c_f, negative steering axis tilt, $-\lambda$, or moving the front center of mass forward away from the steering axis, d. Of course, combinations of these changes in design can also make the bicycle stable, as is illustrated in figures 5.9 - 5.11. Appendix B and C contain the computer program, data, and output verifying the cases discussed in this Chapter.

From the above one might imply that the primitive bicycle with no angular momentum due to wheel spinning can only be self-stable with wheels or added gyroscopes. However, figure 5.12 shows that even a bicycle without wheels can be self-stable. The data associated with this unique bicycle is also shown in Appendix C.

Conclusions

Equations (4.5) and (4.7a) represent the limiting design criteria for a Basic bicycle model, assuming that the model is self-stable at some velocity. These equations are expressed in terms of the bicycle design parameters to ease further research and evaluation of the criteria. The criteria can be used as design guidelines for bicycle models relative self-stability and have no restrictions other than the assumptions needed to create the Basic bicycle model listed in Chapter III of this thesis. Numerous ways exist to use the criteria to gain understanding on bicycle design. Among them are:

1) Vary critical parameters from a standard set of design parameters to optimize the self-stable velocity region. (Make the self-stable velocity as large as possible.)

2) Make the self-stable speed as low as possible.

3) Develop further simplified analytical relations for a bicycle design from these criteria.

4) Create bicycle stability comparison charts using current bicycle products and various size riders.

5) Compare rider handling test results to analytical prediction of stability.

6) Explore radical designs to see if they can be self-stable.

7) Find the effects of controllable parameters on this criterion, by doing trend analysis.

All these ideas for further analysis and research are aimed at increasing understanding of bicycle design.

In this chapter we have demonstrated the use of these criterion to evaluate critically some popular bicycle design guidelines and also to recreate a typical bicycle of today to give our criteria some credibility. The results we have shown seem reasonable.

We emphasize these criteria are only guidelines and the sensitivity to design changes has not been fully explored. The assumptions involved in deriving the criterion make the criteria more applicable to bicycle then motorcycles. However, these criteria are a start towards analytical understanding of two-wheeled vehicles and we hope that in the future they will be even further developed and compared to.

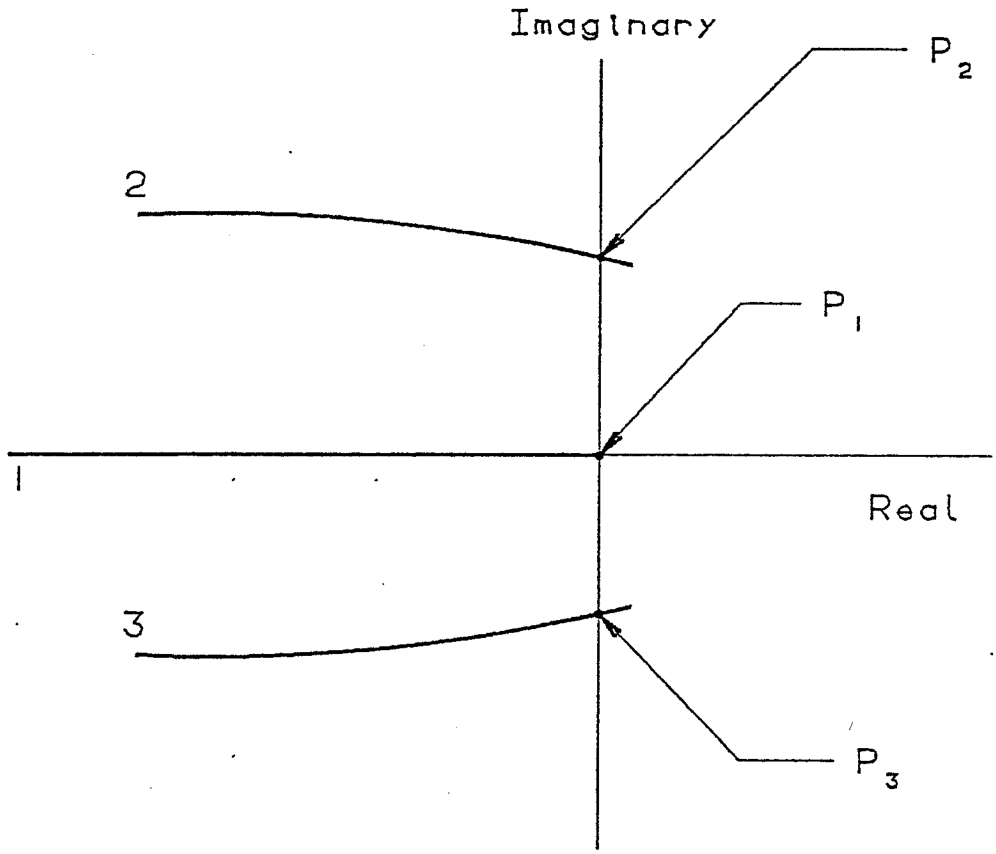
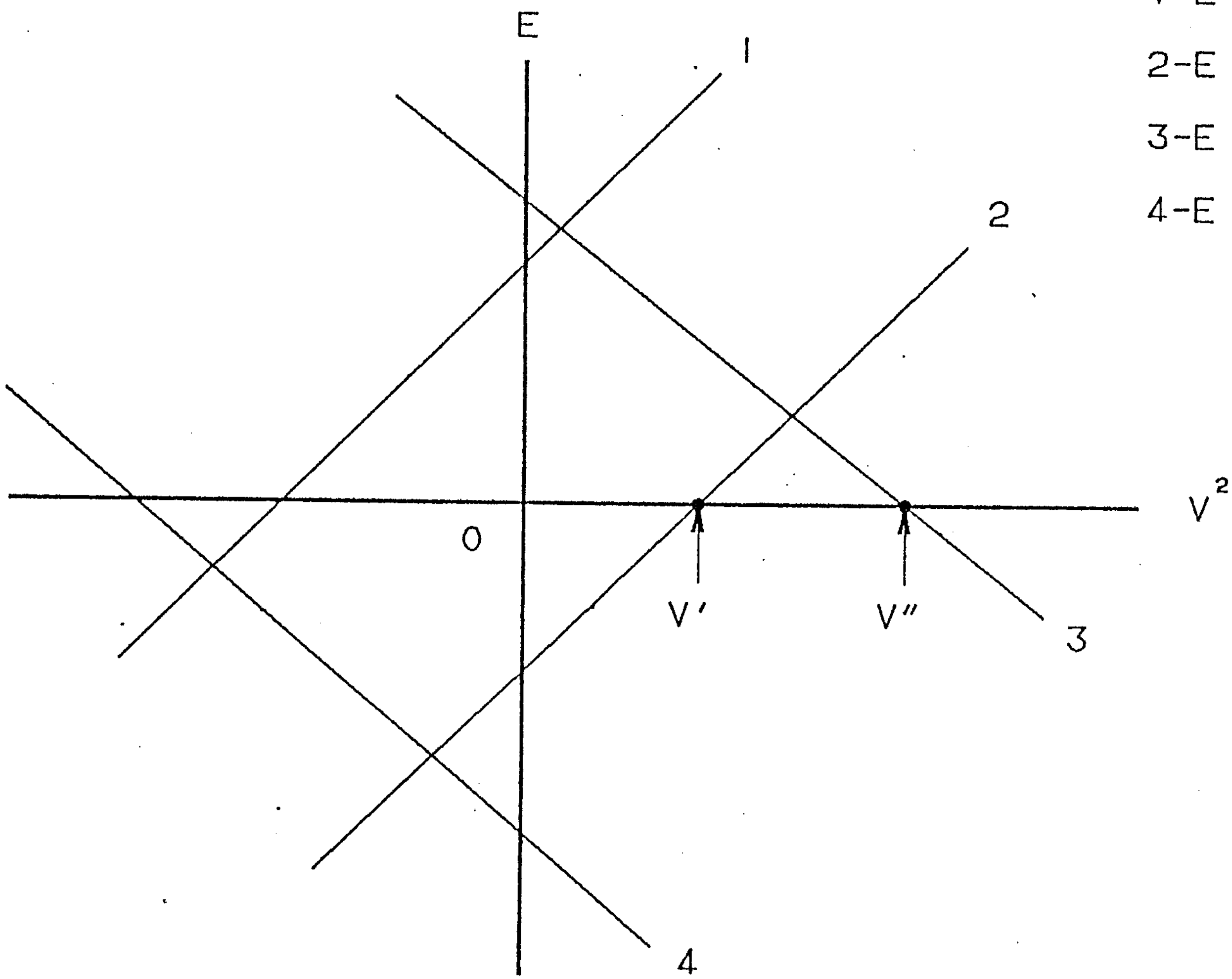


FIGURE 5.1

Ways of going unstable.



1-E criterian always met.

2-E criterian met after V'.

3-E criterian met before V".

4-E criterian never met.

FIGURE 5.2

Possible E criterion configuration plots.

1- 4.7a always met.

2- 4.7a met for $V_6^2 < V^2 < V_5^2$.

3- 4.7a met for $V^2 > V_4^2$.

4- 4.7a met for $V_1^2 < V^2 < V_3^2$.

5- 4.7a met for $V^2 < V_2^2$.

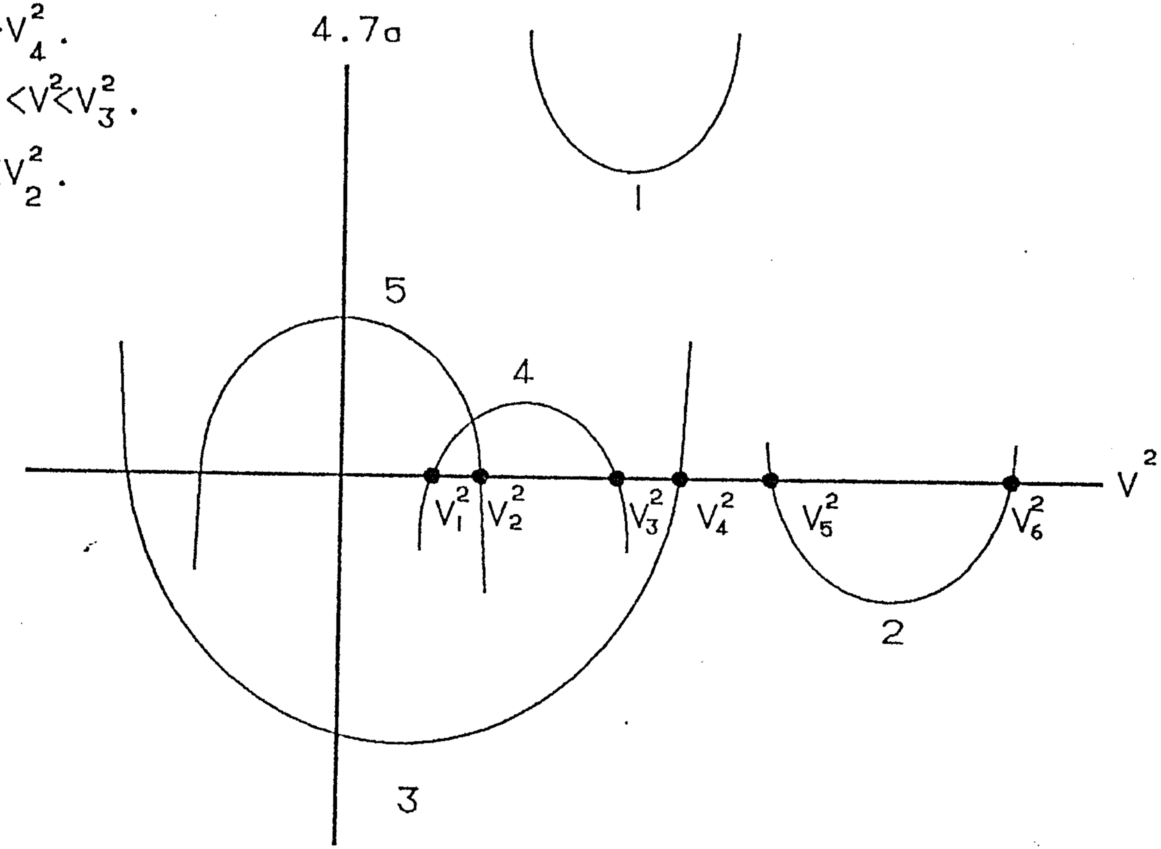


FIGURE 5.3

Possible plots for criterion 4.7a in positive velocity region for which the criterion is positive for some values of V^2.

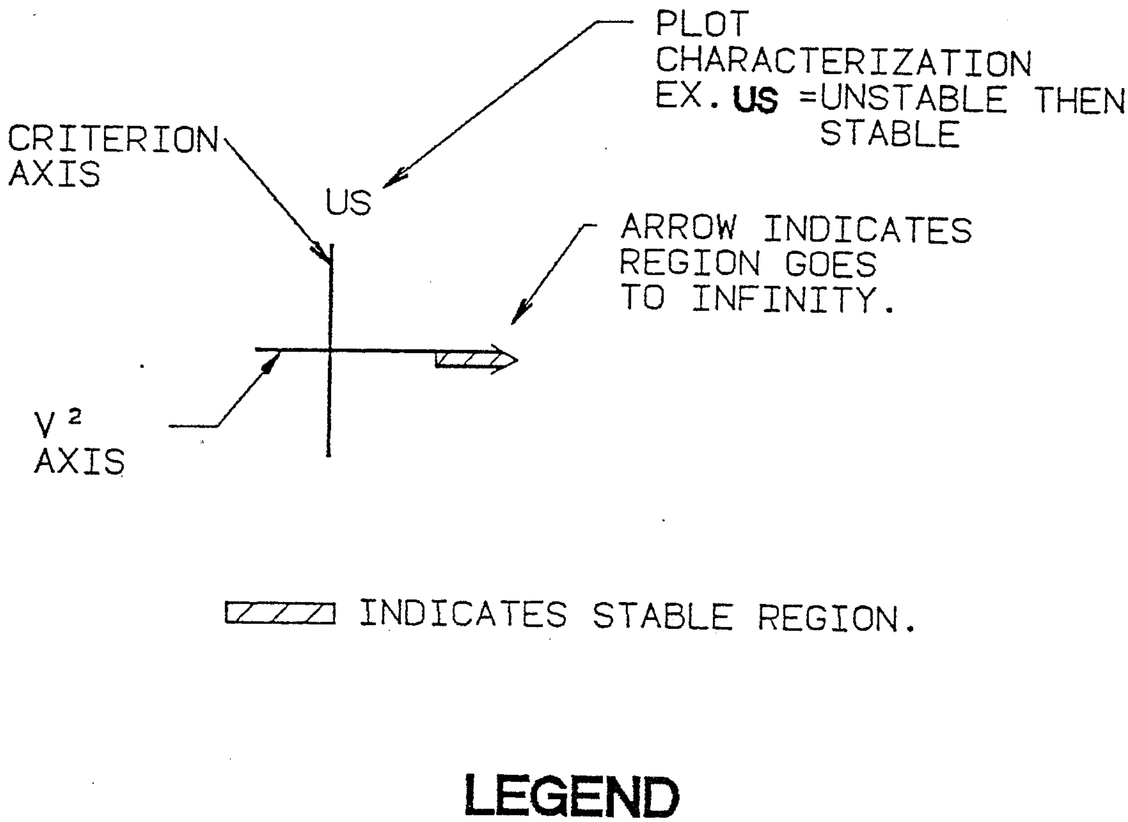


FIGURE 5.4

Possible combinations of criterion E and 4.7a, which define stable regions.

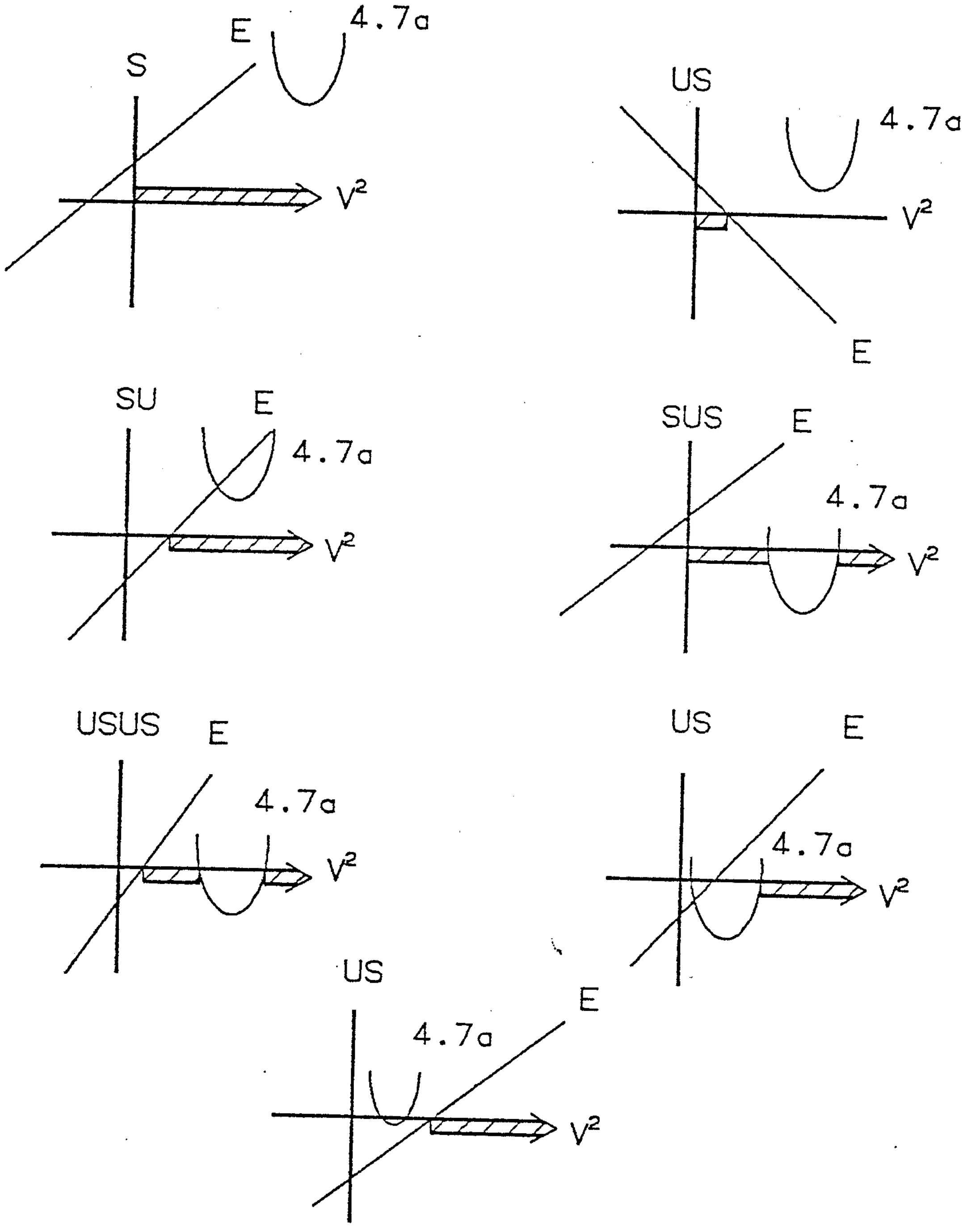


FIGURE 5.4 cont.

Possible combinations of E and 4.7a criterion defining a stability region in the positive V^2 region.

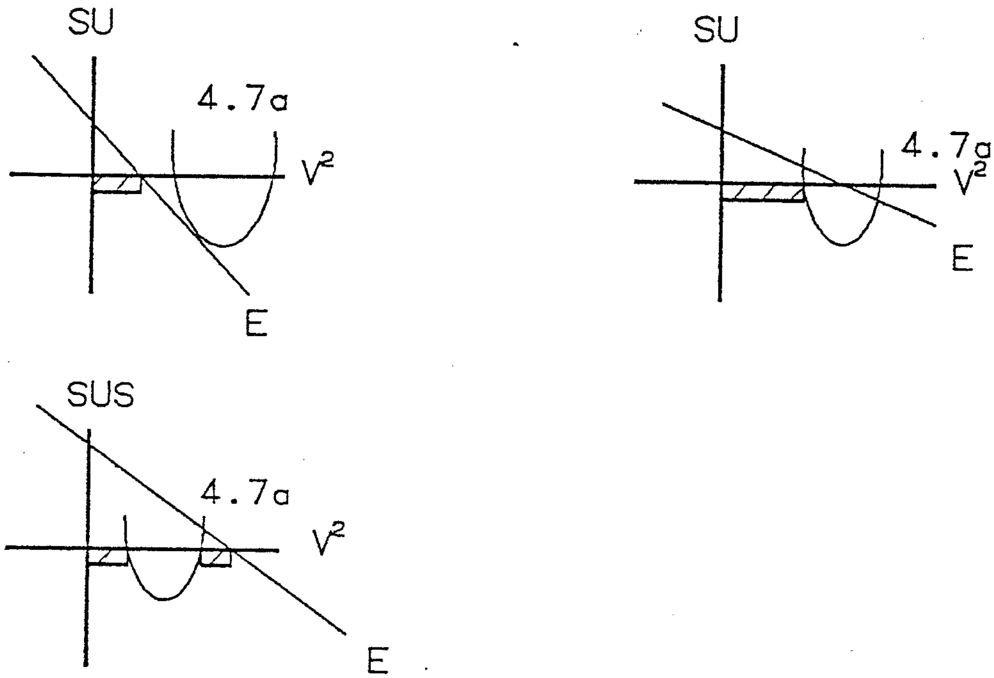


FIGURE 5.4 cont.

Possible combinations of E and 4.7a criterion defining a stability region in the positive V^2 region.

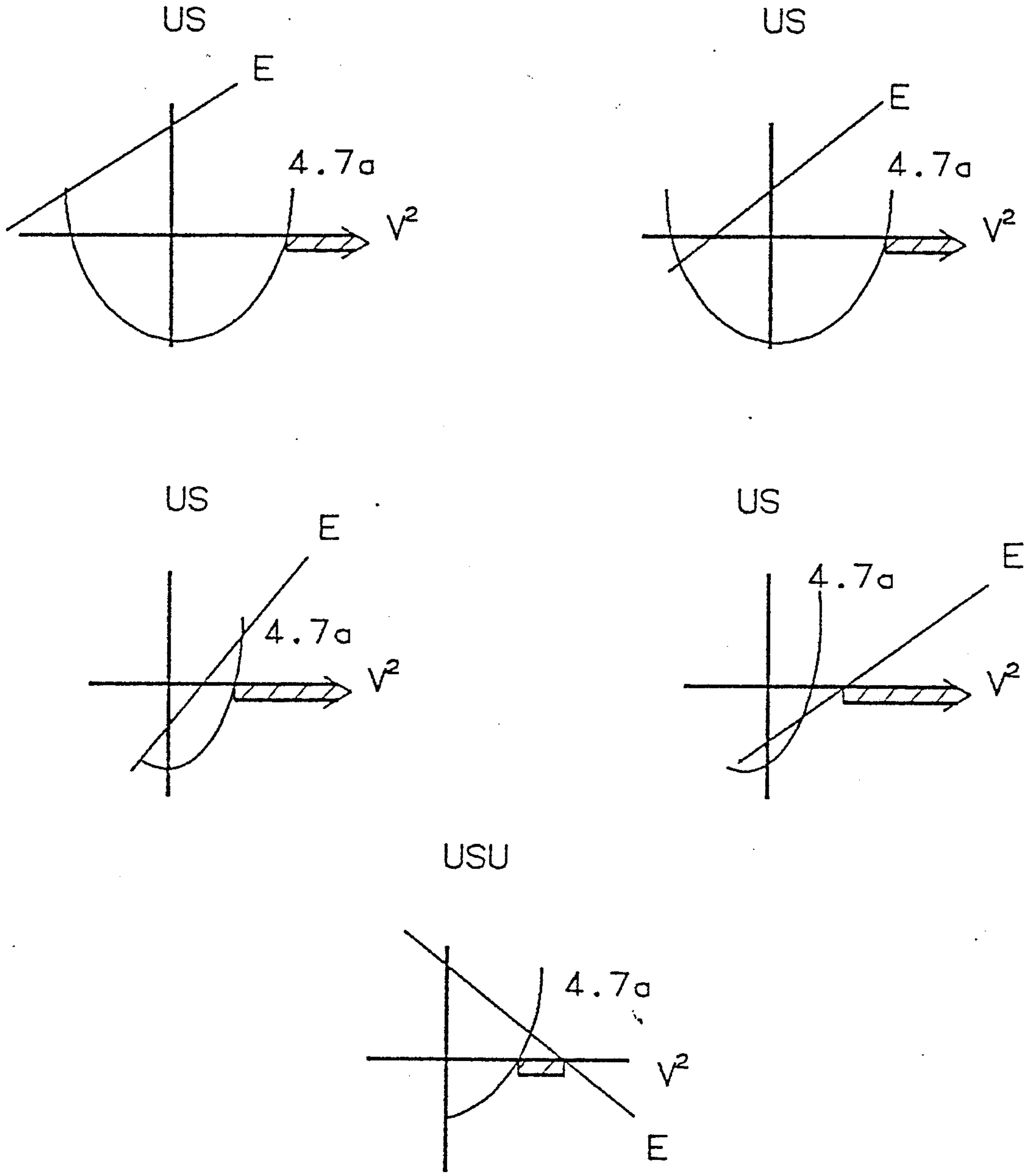


FIGURE 5.4 cont.

Possible combinations of E and 4.7a criterion defining a stability region in the positive V^2 region.

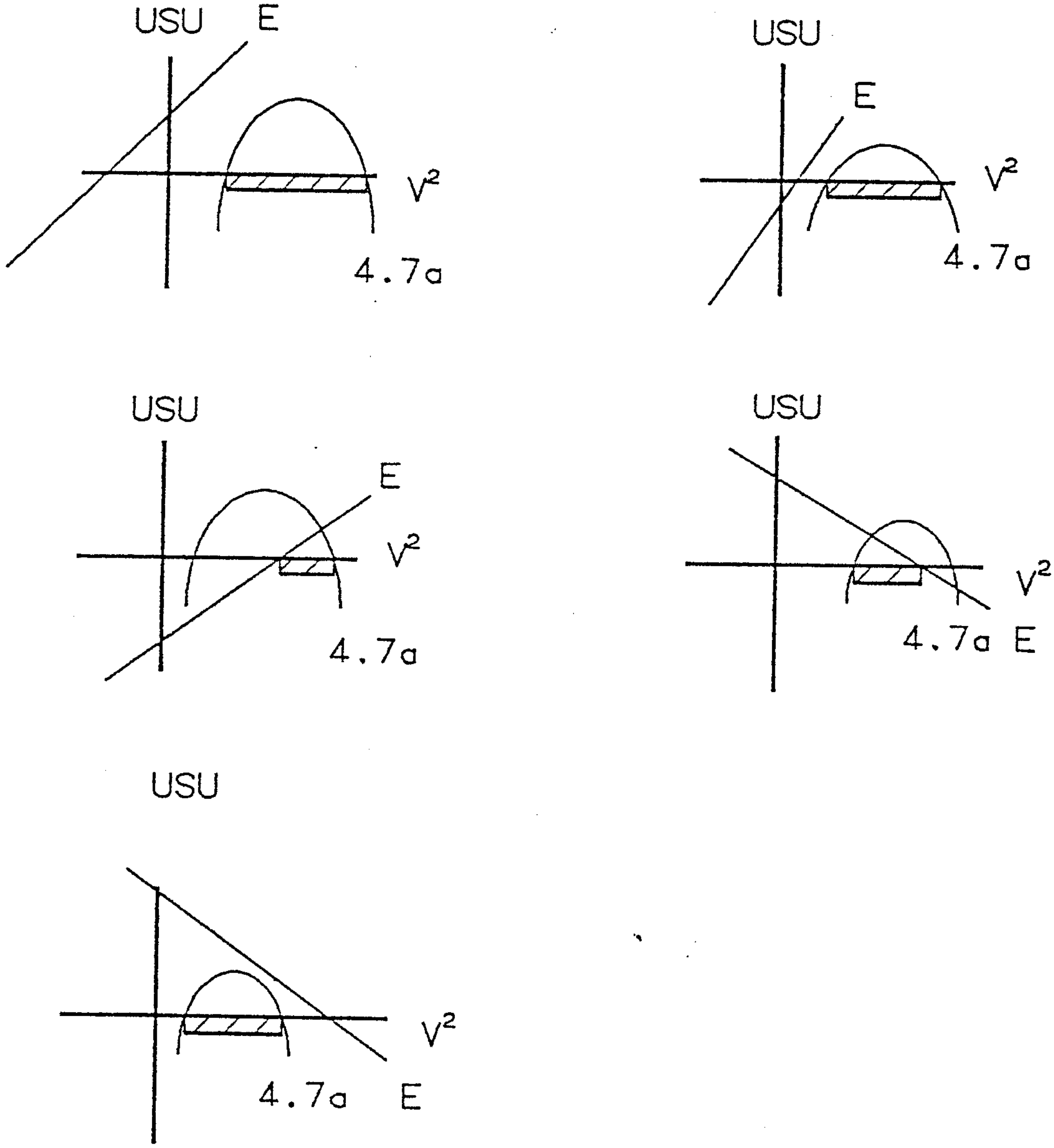


FIGURE 5.4 cont.

Possible combinations of E and 4.7a criterion defining a stability region in the positive V^2 region.

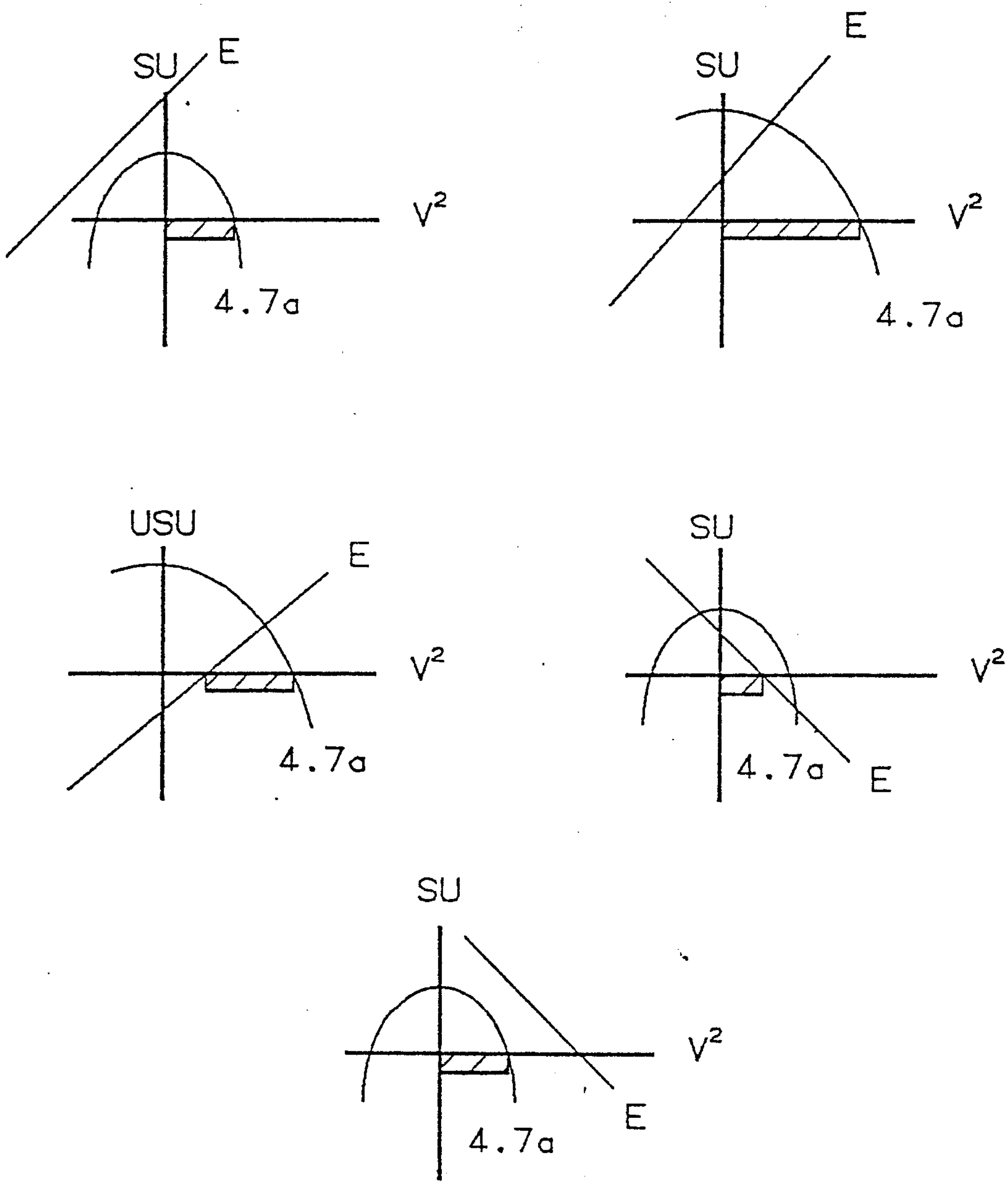


FIGURE 5.4 cont.

Possible combinations of E and 4.7a criterion defining a stability region in the positive V^2 region.

ALWAYS UNSTABLE

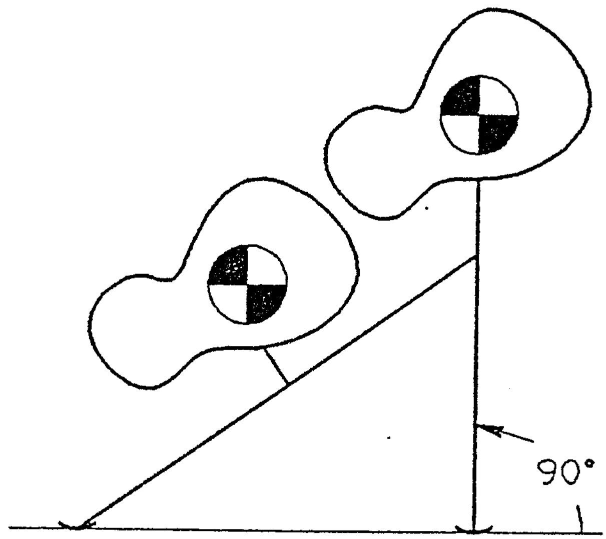


FIGURE 5.5

The primitive bicycle.

ALWAYS UNSTABLE

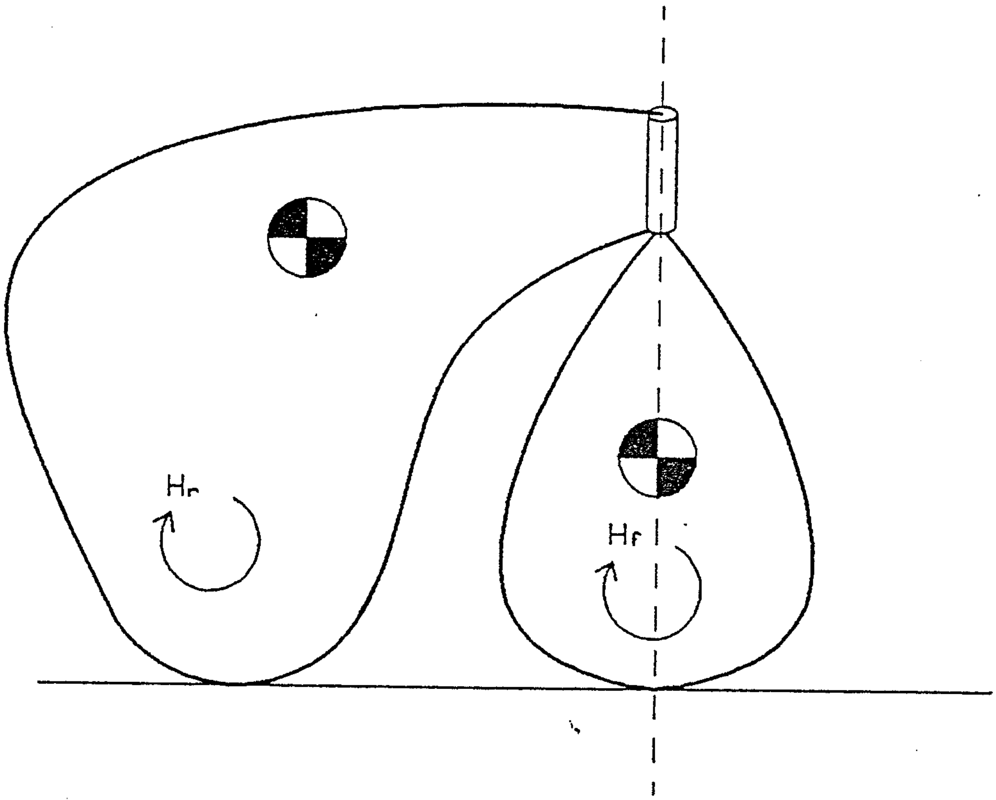


FIGURE 5.6

The primitive bicycle with wheels.

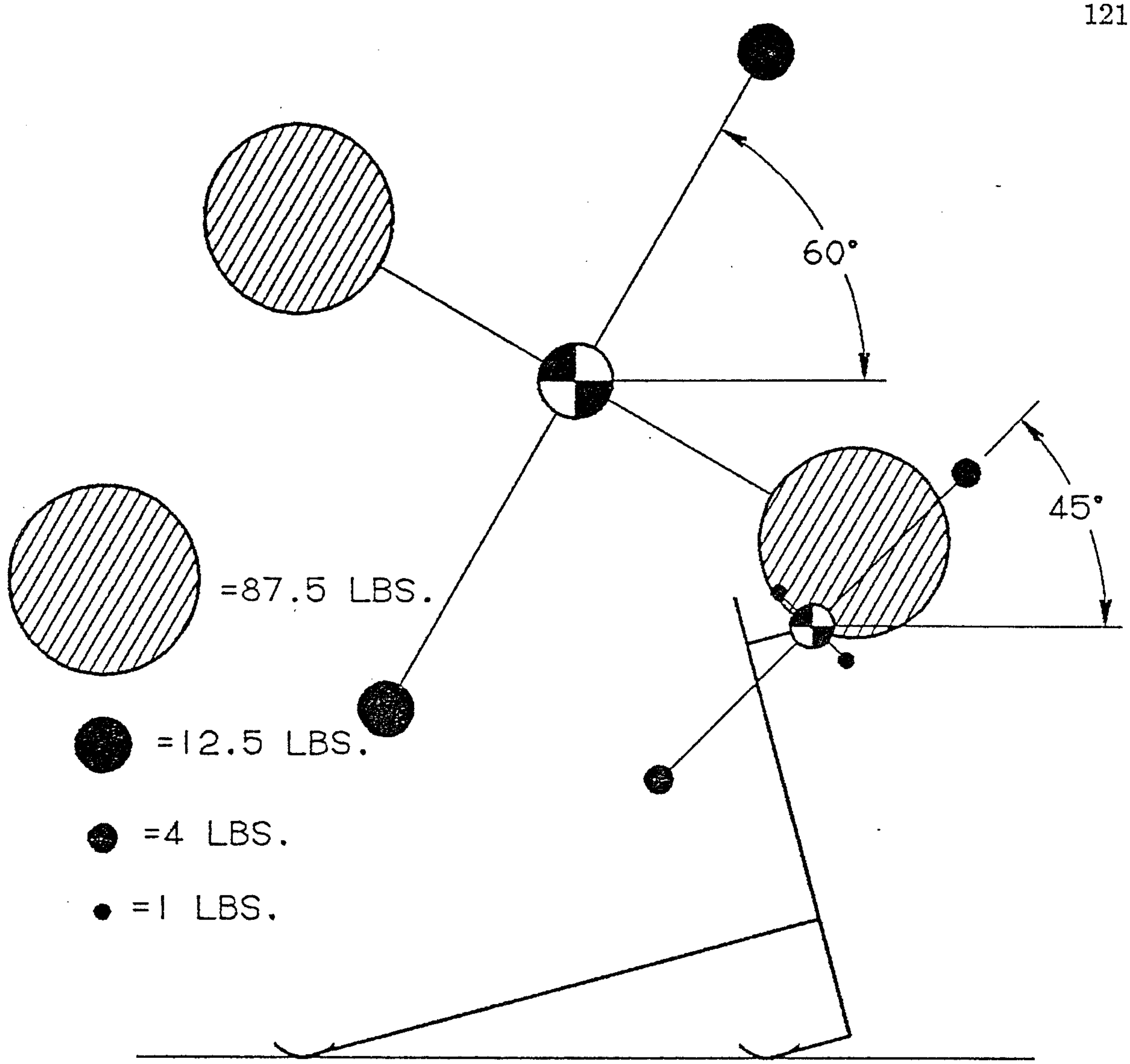


FIGURE 5.7

Primitive model for a rigid rider and bicycle.

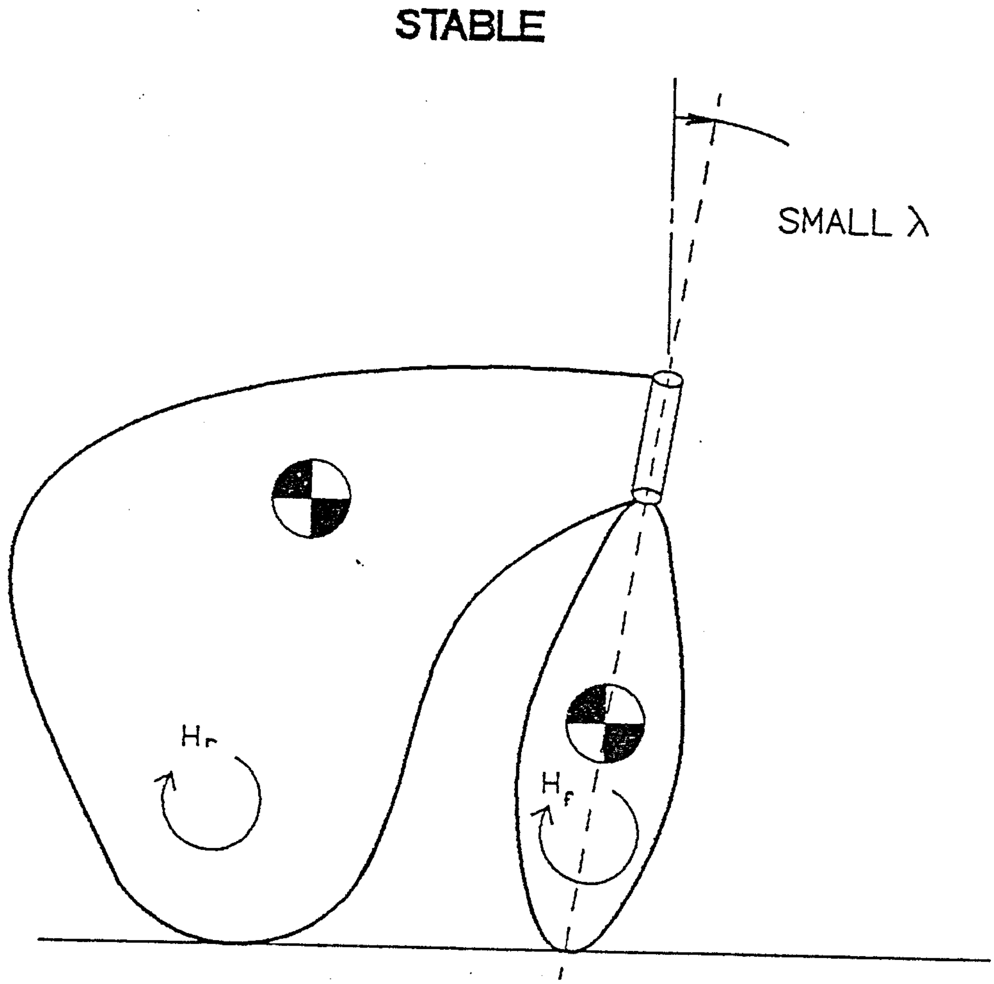


FIGURE 5.8

The primitive bicycle with wheels and a slight negative tilt of the steering axis.

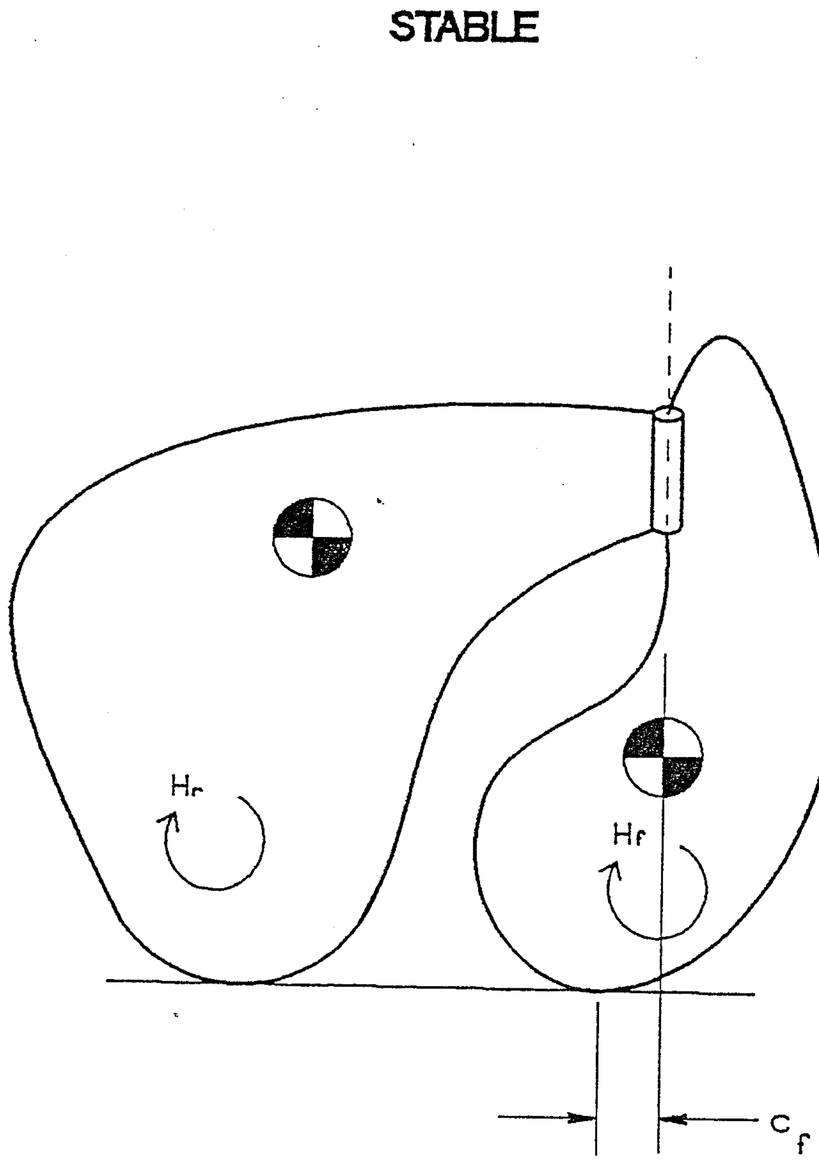


FIGURE 5.9

The primitive bicycle with wheels and trail.

STABLE

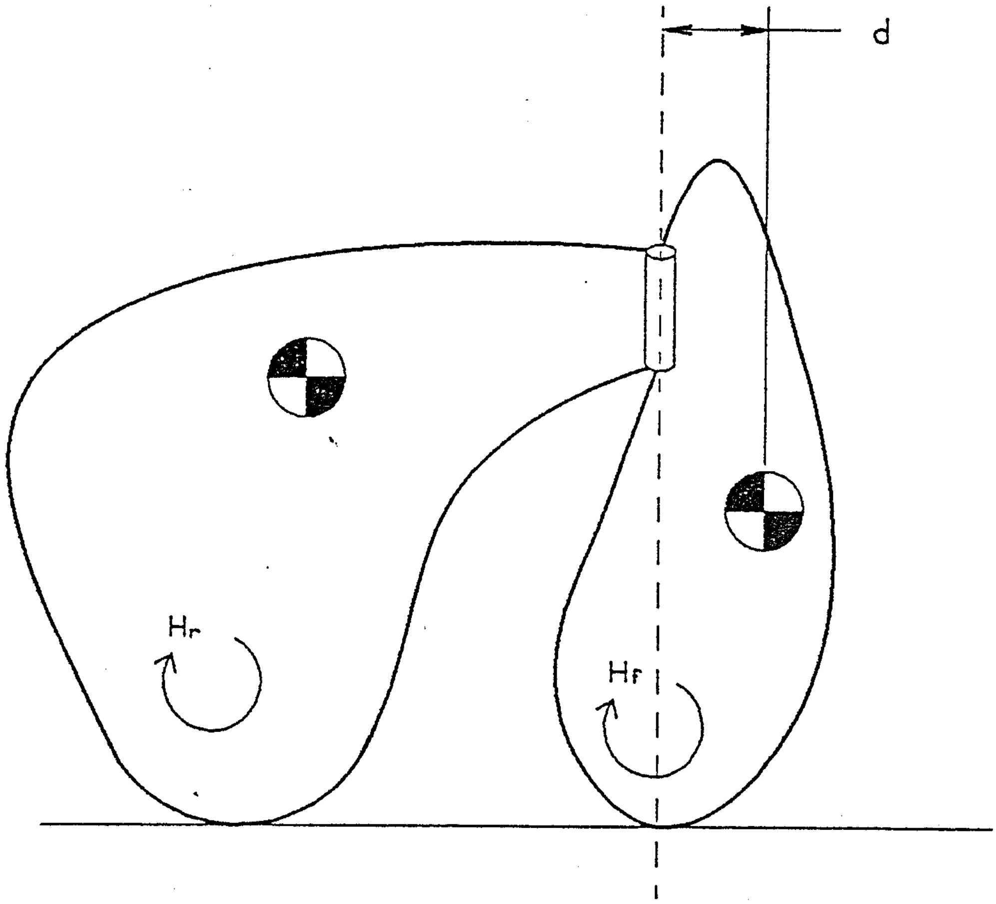


FIGURE 5.10

The primitive bicycle with wheels, and front center of mass offset from steering axis.

STABLE

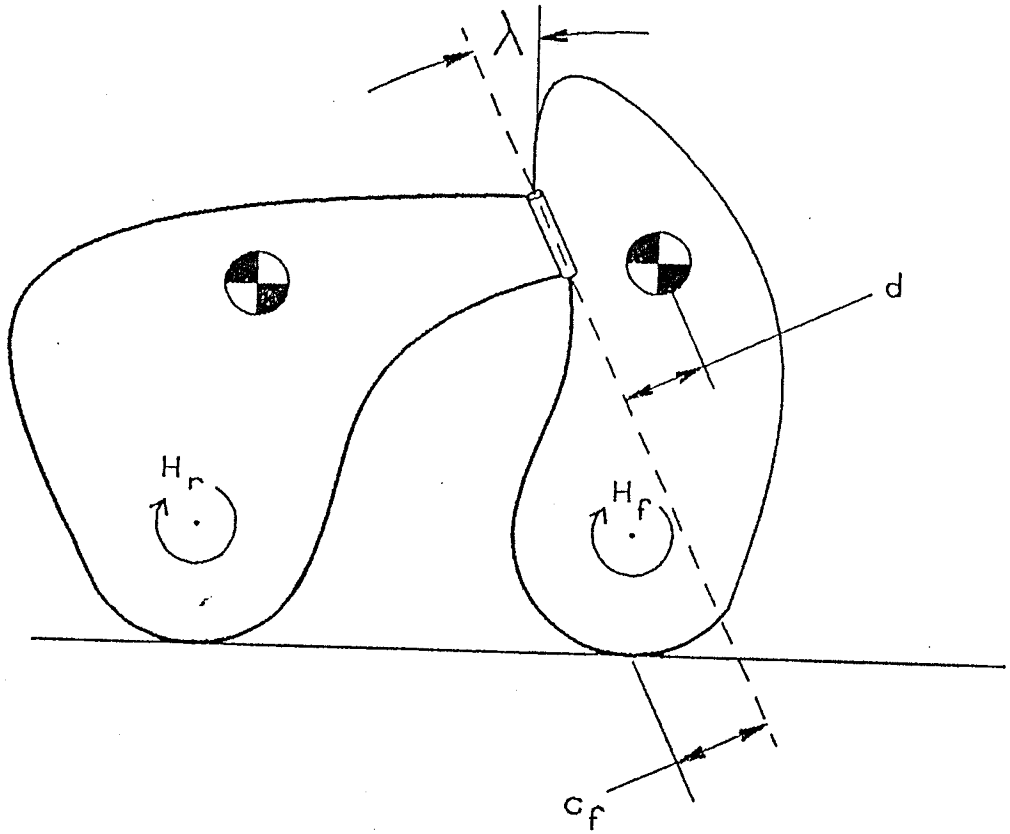


FIGURE 5.11

A stable bicycle model using data approximating a Schwinn Varsity bicycle and rider.

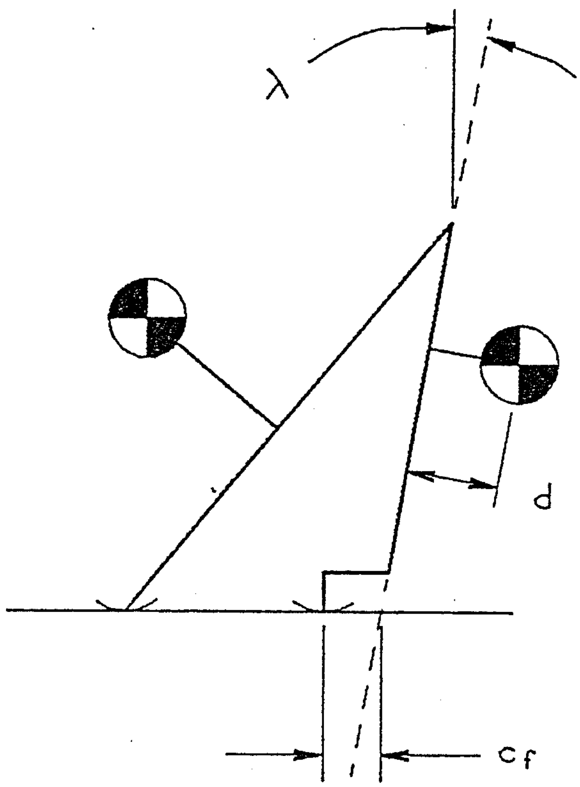


FIGURE 5.12

A stable bicycle model with no wheels.

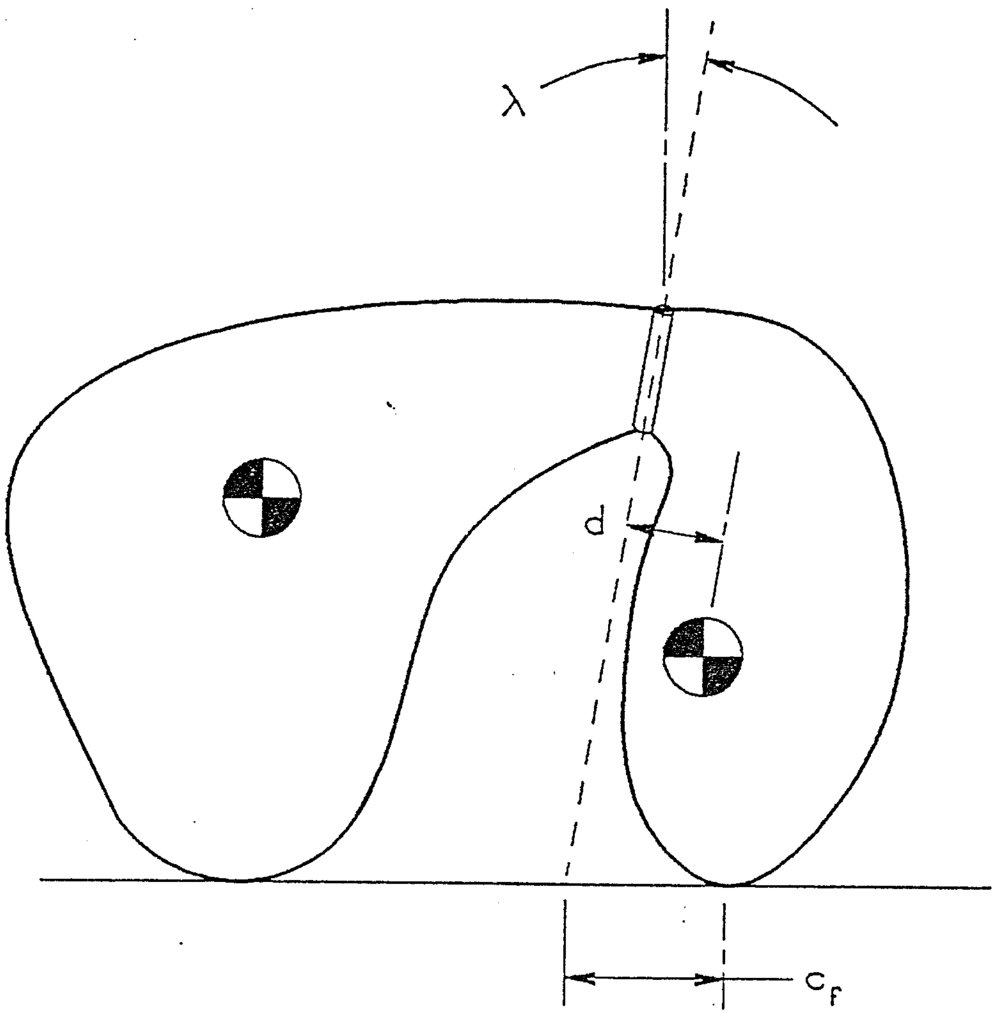
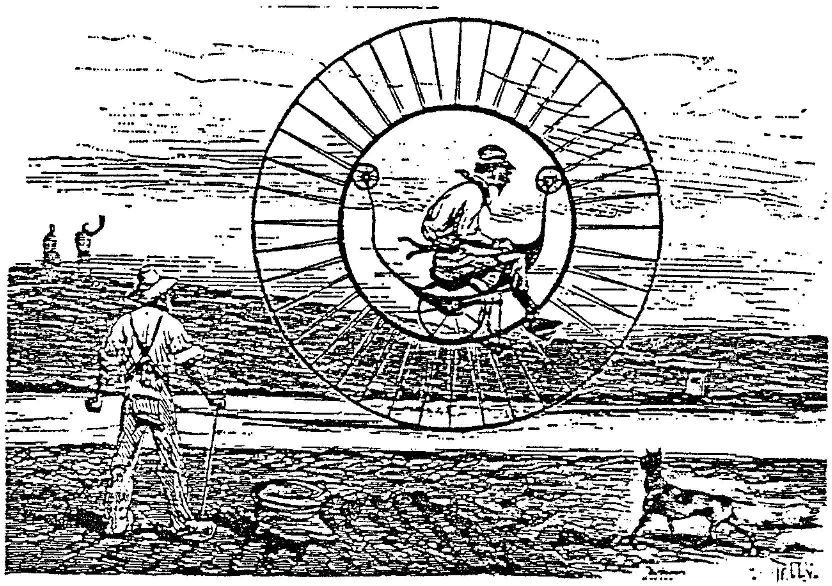


FIGURE 5.13
A stable bicycle model with negative trail.



VI. SUMMARY OF CONCLUSIONS AND RECOMMENDATIONS

Summary of Conclusions

In Chapter III of this thesis we defined a Basic bicycle model. We then derived the linearized equations of motion for the Basic bicycle model using Lagrange's equations. The Lagrangian formulation of the equations was chosen because it was felt an easier method to apply, requiring fewer algebraic steps. The final two governing equations (3.14-15) appear at the end of Chapter III. These equations are in considerably simpler form when compared to most past works. The two equations are functions of seventeen independent design parameters shown in figures 3.5a-b. The equations are in general form and can be used for analyzing bicycle self-stability or control studies.

In Chapter IV we made an extensive comparison of our equations to twenty other sets of equations of motion presented in past works. In all, only three of the twenty papers compared had equations equilvalent to ours. We found that the earliest published derivation known for a Basic bicycle model (Whipple [1899]) is in agreement with our linearized equations except for some typos. Only two derivations (Döhring [1955] and Weir [1972]) were as general as ours and had equations that agree completely. Singh and Goel [1971] used Döhring's [1955] equations. Other authors did not have as general a model, made algebraic mistakes in their derivation, had typos, or disagreed in other ways. We note that Döhring [1955] and Weir [1972] were both Newtonian derivations and ours Lagrangian.

In Chapter V we developed the Routh-Hurwitz criteria for the Basic bicycle model's equations of motion to study bicycle self-stability. Numerically evaluating the design criteria using a computer we found several self-stable configurations. Although no general analytical conclusions were made, numerical evaluation of the criteria supports analytical simplifications.

After expanding these criteria, because they are complicated, we applied them to a 'primitive' model showing that the parameters of steering axis tilt, mechanical trail, and front center of mass offset have immediate impact on the self-stability of the bicycle. These parameters significantly changed the size and location of the stable velocity region for the bicycle designs we explored. Other parameters could have similar effects but were not investigated. Just for fun, at the end of Chapter V we included a self-stable bicycle design which has no spinning angular momentum due to the wheels and a self-stable bicycle design which has negative trail. The numerical values used for the parameters in these models are contained in Appendix C.

In conclusion, we feel that the design criteria developed in this thesis can be used to improve bicycle design and better understand the effect of design changes on bicycle stability. We remind the reader that the premise on which this conclusion is based, is that *short term* rider controlled stability is closely linked to the bicycle's *self-stability.* We believe the premise is valid because in the short term the rider has little control of the bicycle. Hence, design change effects on the bicycle's self-stability will have almost immediate impact on the rider's prespective of handling, and possibly performance. We hope that the design criteria developed in this thesis will be investigated further and the computer program enhanced for designers to study the stability characteristics of their designs.

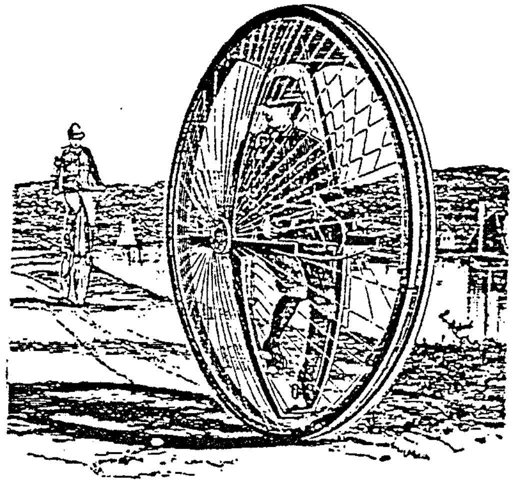
Recommendations

Because we were able to quickly and easily generate several self-stable rider-bicycle configurations, we feel that there is potential that many other stable rider-bicycle configurations exist. Based on the results of stable configurations found in Chapter V, we feel there is no reason to limit stability studies to only consider small modifications to current designs (which is what we observed in most papers referenced). We recommend that new, possibly radical, configurations be studied. These new configurations could be compared to popular designs. New bicycle designs could be safer, give more preferred handling characteristics, and, possibly improve performance.

Because using Routh-Hurwitz criteria lacks in giving information about how long a perturbation will take to decay, we recommend the Routh-Hurwitz criteria be used in combination with eigenvalue-eigenvector studies to further develop analytical guidelines for bicycle design. These guidelines could be modified for *Augmented* bicycles having added gyros, springs, and dampers to see their effect analytically and numerically on bicycle stability. The guidelines could also be developed after prescribing a controller, $\psi(\theta_r)$, if wanted.

Based on the results given in Chapter V, the evolution of bicycle design is not clear. Popular bicycle designs have not evolved into the "most stable" design configuration, at least from a self-stability standpoint. This implies that the rider may not desire the most stable configuration, or that there might be some design configurations that for reasons we still don't understand are more desirable. For this reason we recommend that the popular assumption that the bicycle be designed to be as stable as possible (real eigenvalues as negative as possible) be reviewed. It may well be that riders do not prefer an analytically self-stable bicycle at all speeds.

Finally, a method for measuring a rider's desires in bicycle stability is needed to compare to theoretical results. Hopefully, this will enable bicycle builders to more readily customize bicycle designs to a rider preferred stability configuration.



APPENDIX A

SPECIFIC DETAILS, THEORETICAL JUSTIFICATION, AND PHYSICAL INTERPRETATION OF THE DERIVATION OF THE LINEARIZED EQUATIONS OF MOTION

Section 1: Derivation of Relations Between the Generalized Coordinates and the Auxiliary Variables

In chapter III the relationships between the generalized coordinates and the auxiliary variables for small angles were approximated as,

$$X_f - X_r = -c_w\theta_r + c_f\psi \tag{3.1a}$$

$$Y_f - Y_r = c_w \tag{3.1b}$$

$$\chi_f - \chi_r = -\psi \sin\lambda \tag{3.1c}$$

$$\theta_f - \theta_r = \psi \cos\lambda \tag{3.1d}$$

The first two of these equations, (3.1a) and (3.1b), are essentially an expression of the front contact point relative to the rear contact point in the ground plane. The second pair of equations, (3.1c) and (3.1d), are the angles which describe the front fork/handlebar assembly relative to the rear frame; that is, between the body fixed axes $\overline{x}_r\overline{y}_r\overline{z}_r$ and $\overline{x}_f\overline{y}_f\overline{z}_f$. This section shows how to derive these relations.

The relations can be derived in two ways. The first is the more formal method which involves deriving nonlinear equations relating the auxiliary variables X_f, Y_f, χ_f, θ_f to the generalized coordinates, X_r, Y_r, χ_r, and θ_r, then linearizing the equations about the vertical equilibrium position. This method involves several coordinate transformations and proved to be time consuming, tedious and as far as the author knows, not known in closed form.[1]

The second method involves approximating the displacements of the front contact relative to the rear contact and treating small angles of rotation as vectors. This method yields the same linearized results as the nonlinear approach should. It, however, is mathematically simpler and physically easier to understand.

What follows is a description of how to derive equations (3.1$a-d$) using the second method. The notation used here is the same as in chapter III.

First, we derive equations (3.1a) and (3.1b) which relate the rear and front contact points. To do this, we approximate the displacements due to small an-

[1] See Psiaki [1979]. We should note that although Nĭemark and Fufaev's [1967] nonlinear equations yeild the correct expressions to first order, their equations neglect the rise and fall of the bicycle due to steering effects. And hence, are not the correct nonlinear kinematic equations as presented.

gles of θ_r and ψ individually, and then add their displacements to get the linear approximation.

Figure A.1 shows a top view of the track of the bicycle with no steer but a small angle θ_r. The difference between X_r and X_f to first order due to this positive rotation θ_r is $-c_w\theta_r$. Now, adding a positive angle of rotation due to the steer ψ, as shown in figure A.2, the difference between X_r and X_f is approximated to first order as,

$$X_f - X_r \simeq -c_w\theta_r + c_f\psi \tag{A.1a}$$

where the term $+\ c_f\psi$ is seen by looking down the steer axis, holding θ_r constant, and approximating the sideways motion for a positive steer to first order as shown in figure A.3.

The difference between the front contact point and rear contact point in the Y direction to first order is simply the wheelbase length,

$$Y_f - Y_r = c_w \tag{A.1b}$$

However, because $\dot{Y}_r$ has a nonzero equilibrium position in the derivation, the difference between Y_f and Y_r is needed to second order when calculating $\dot{Y}_f$, which is used in the kinetic energy equations. Adding the second order effects due to yaw and steer, as shown in figure A.4 we get,

$$Y_f - Y_r = c_w\left(1 - \frac{\theta_r^2}{2}\right) + c_f\frac{\psi^2}{2}\cos\lambda$$

In chapter 3 the above equation was designated $(3.1b)'$. As it turns out the higher order terms in the θ_r and ψ, although technically required, do not contribute to Lagrange's equations.

Next, to derive equations (3.1c) and (3.1d), we again treat small angles of rotations as vectors. This time rather than approximating small displacements, we approximate small rotations. This approach is identical to adding the angular velocity vectors for relative motion of rigid bodies.[2] It is emphasized that this can only be done for the first order approximation.

Applying this to the Basic bicycle model shown in figure A.5, this means that the angle of the rear frame (rear body fixed coordinates $\overline{x}_r\overline{y}_r\overline{z}_r$) from its equilibrium position with respect to the inertial reference XYZ, $\gamma_{r/XYZ}$, plus the angle of the front fork/handlebar assembly (front body fixed coordinates $\overline{x}_f\overline{y}_f\overline{z}_f$) relative to the rear frame, $\gamma_{f/r}$, yields approximately the angle of the front fork/handlebars relative to the inertial frame of reference, $\gamma_{f/XYZ}$. This is shown in figure A.5. This approximation becomes exact in the limit as the small angles of rotation go to zero. It is correct however, to first order.[3] Using the approximation we can express this relation in vector form as,

$$\vec{\gamma}_{r/XYZ} + \vec{\gamma}_{f/r} \simeq \vec{\gamma}_{f/XYZ} \qquad (A.2a)$$

For our Basic bicycle model $\vec{\gamma}_{f/r}$ is $\psi\vec{\lambda}$, where $\vec{\lambda}$ is the unit vector positive up along the steering axis and ψ is the magnitude of $\vec{\gamma}_{f/r}$.

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[2] See Meriam and Kraige [1986] pp. 314.

[3] See Shames [1980].

Expressing the vectors $\vec{\gamma}_{r/XYZ}$ and $\vec{\gamma}_{f/XYZ}$ in the $\overline{x}_r, \overline{y}_r, \overline{z}_r$ and $\overline{x}_f, \overline{y}_f, \overline{z}_f$ coordinate systems respectively, we can rewrite equation $(A.2a)$ to first order as follows,

$$\theta_r \vec{\mathrm{K}} + \chi_r \vec{\mathrm{j}}_r + \psi \vec{\lambda} \simeq \theta_f \vec{\mathrm{K}} + \chi_f \vec{\mathrm{j}}_f \qquad (A.2b)$$

However, for small angles,

$$\vec{\mathrm{j}}_f \simeq \vec{\mathrm{j}}_r \simeq \vec{\mathrm{J}}$$

and,

$$\vec{\lambda} \simeq \cos\lambda \vec{\mathrm{K}} + \sin\lambda \vec{\mathrm{J}}$$

These relations can be proven to be correct to first order for small angles of θ_r, ψ, and χ_r, by writing out $\vec{\mathrm{j}}_r$, $\vec{\mathrm{j}}_f$, and $\vec{\lambda}$ as functions of $\vec{\mathrm{J}}$, $\vec{\mathrm{I}}$, and $\vec{\mathrm{K}}$.[4]

Writing equation (A.2b) in terms of the inertial coordinates system's unit vectors we have,

$$\theta_r \vec{\mathrm{K}} + \chi_r \vec{\mathrm{J}} - \psi \sin\lambda \vec{\mathrm{J}} + \psi \cos\lambda \vec{\mathrm{K}} \simeq \theta_f \vec{\mathrm{K}} + \chi_f \vec{\mathrm{J}} \qquad (A.2c)$$

equating the $\vec{\mathrm{K}}$ and $\vec{\mathrm{J}}$ components we get two scalar equations equivalent to (3.1c) and (3.1d)

$$\chi_r - \psi \sin\lambda \simeq \chi_f \qquad (A.1c)$$

$$\theta_r + \psi \cos\lambda \simeq \theta_f \qquad (A.1d)$$

Thus, equations (A.1a-d) agree with (3.1a-d) presented in chapter III.

[4] This may not be possible in closed form.

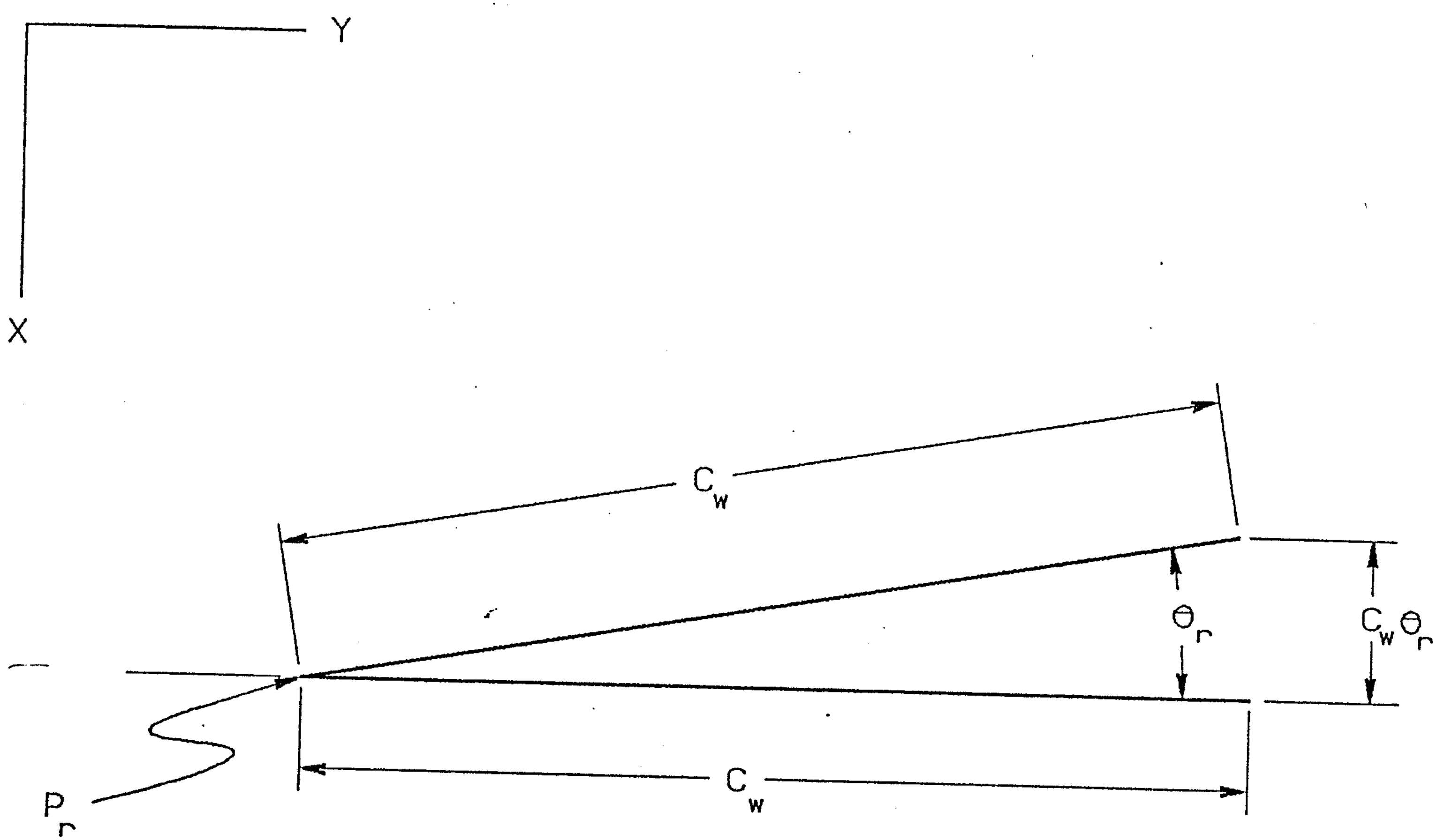

FIGURE A.1
Top view of the track of the bicycle with no steer angle, Ψ, but a small yaw angle, Θ_r.

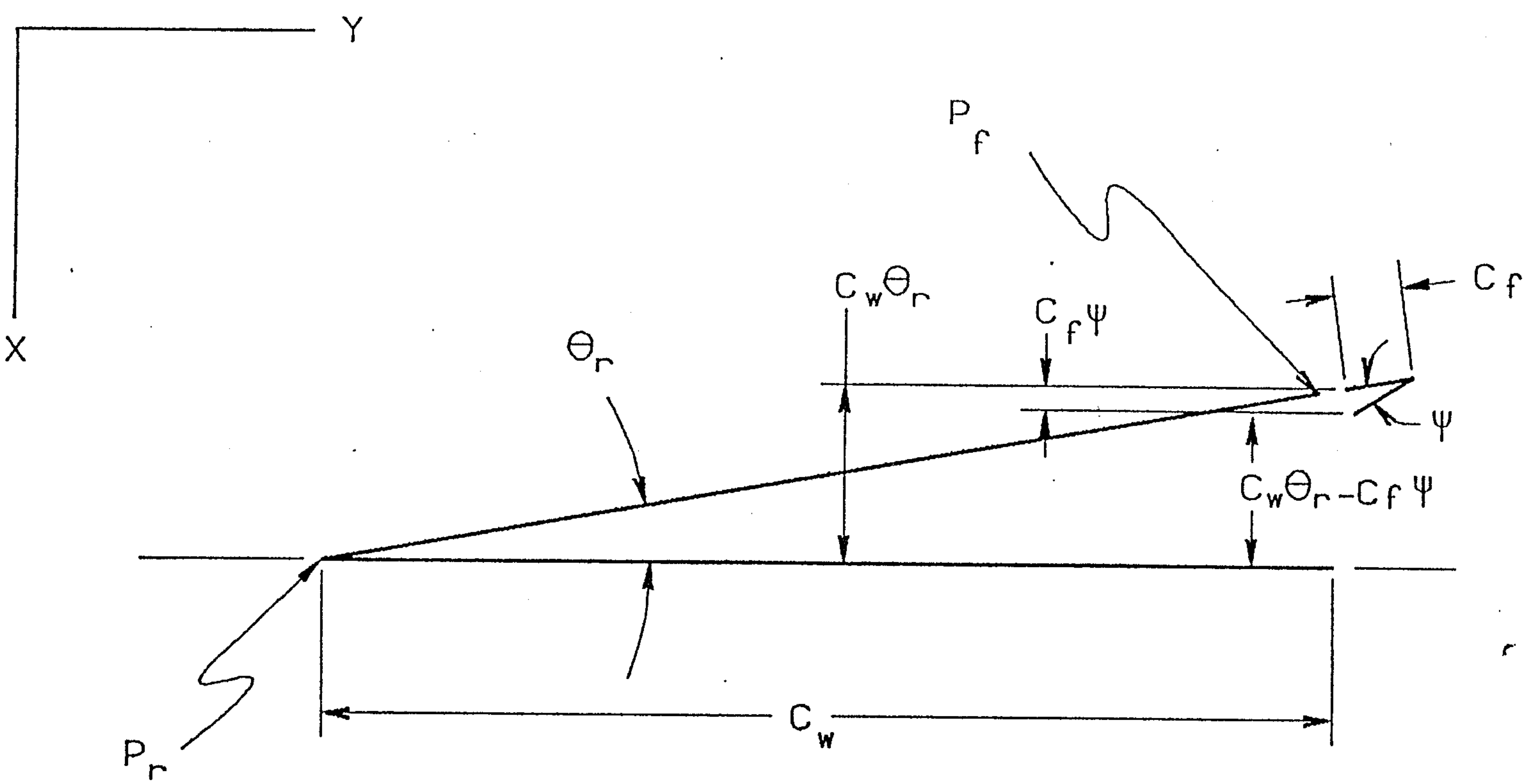

FIGURE A.2

Top view of the track of the bicycle with small steer, Ψ, and small yaw angle, Θ_r.

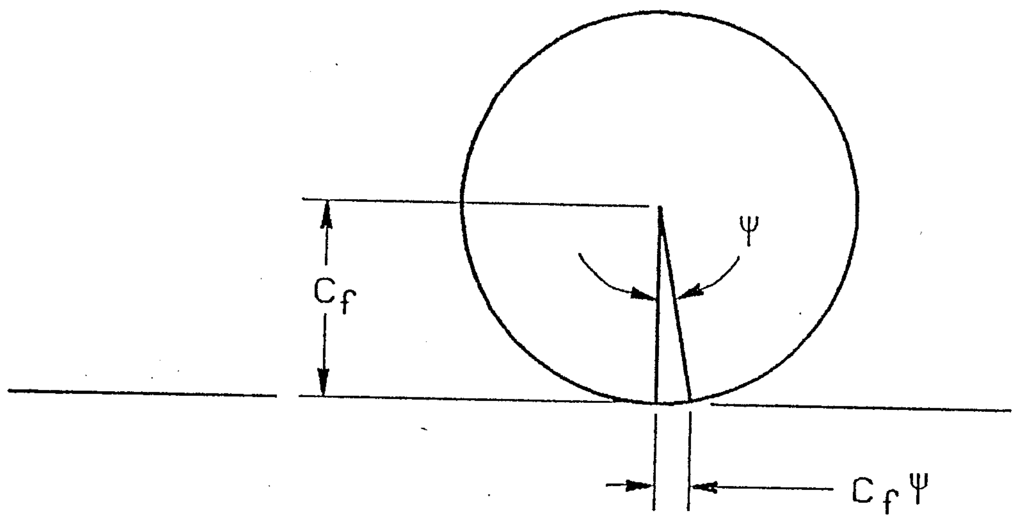

FIGURE A.3

Sideways motion to first order of the front wheel contact for a small positive steer angle.

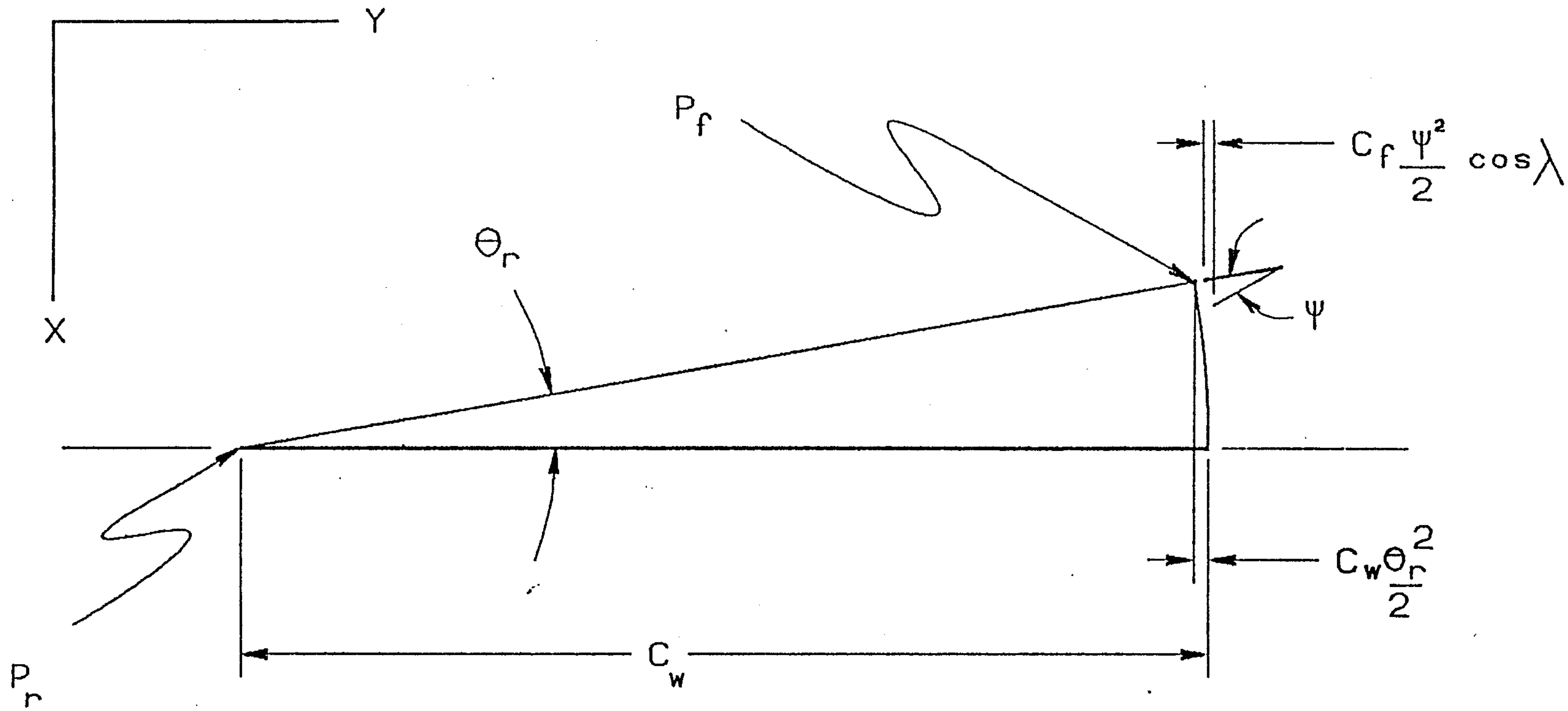

FIGURE A.4

Some second order effects due to yaw angle, Θ_r, and steer angle, Ψ, on front wheel contact position.

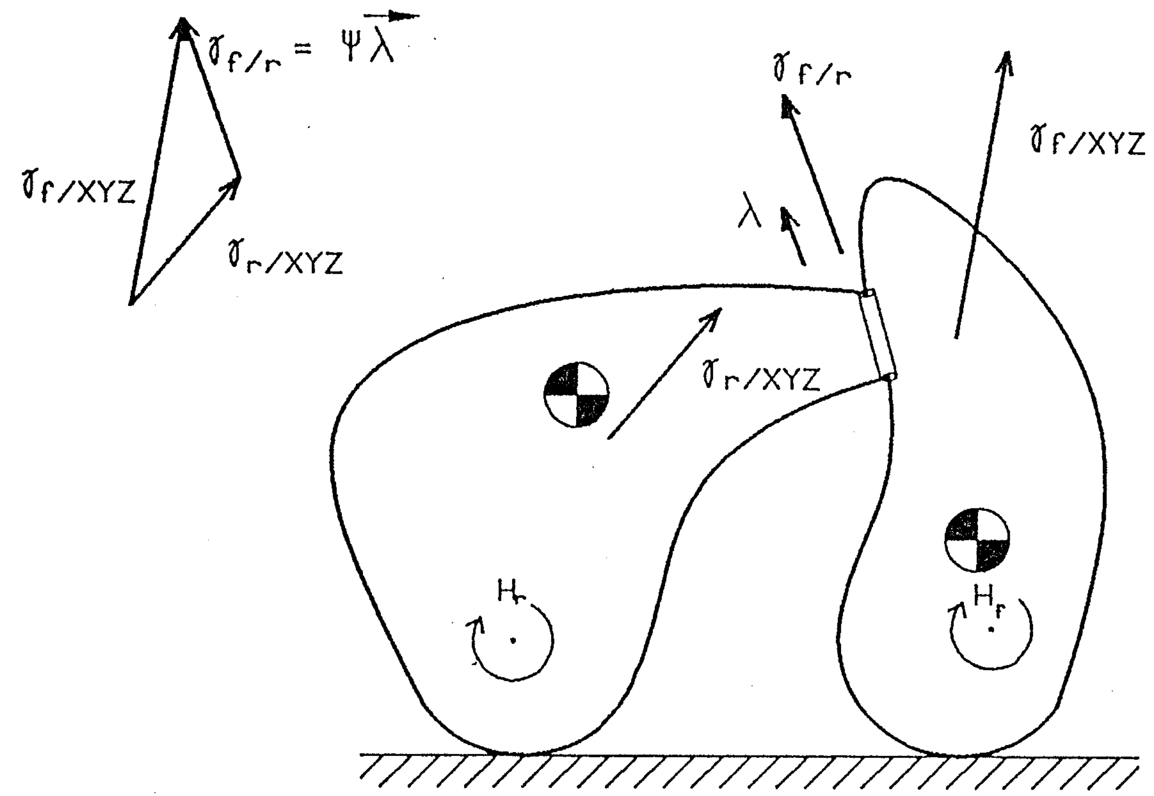

FIGURE A.5

The basic bicycle model with small orientation angles treated as vectors, $\vec{\gamma}_{r/XYZ}$, $\vec{\gamma}_{f/r}$, $\vec{\gamma}_{f/XYZ}$.

Section 2: Derivation of Constraint Relations for the Basic Bicycle Model

In chapter III, equations (3.2a-e) were used to simplify the Lagrange's equations and said to be derived from the constraint relations on the front and rear contact points motion for the Basic bicycle model. This section shows how the derivation of equations (3.2a-e) follows from simplifying the constraints that exist on a bicycle with thin rigid disks as wheels.

For the Basic bicycle model we have assumed the tires to be part of the wheels which are assumed thin rigid disks. This implies infinitely stiff tires, so no side-slip angle can exist on our Basic bicycle model. That is, the direction of the velocity of the contact point is defined by the intersection of the plane of the wheel and the ground plane. This direction is referred to as the instanteaous direction that the wheel is headed.

In addition, we assume enough friction exists between the thin rigid disks and the ground so that there is no relative motion between the point of contact of the rigid wheel and the ground. That is, there is no sliding of the wheel on the surfaces of the road. In the practical sense this could be caused by oil on the pavement or loose gravel.

Based on the above assumptions, four nonlinear kinematic rolling constraints exist for the Basic bicycle model. These constraints are nonholonomic and can be added to the problem only after developing Lagrange's equations.[5] The constraint

[5] This is what the author interprets from Goldstein [1980].

equations relate the velocity of the rear and front contact point velocity to their respective heading in the ground plane and respective wheel rotations. Writing these in their nonlinear form for the rear contact point velocity we have,

$$\dot{Y}_r = a_r \dot{\phi}_r \cos\theta_r \tag{A.3a}$$

$$\dot{X}_r = -a_r \dot{\phi}_r \sin\theta_r \tag{A.3b}$$

where $\dot{\phi}_r$ is the angular velocity of rotation of the rear wheel in its own plane, that is, the spin rate. Similarly for the front contact point,

$$\dot{Y}_f = a_f \dot{\phi}_f \cos\theta_f \tag{A.3c}$$

$$\dot{X}_f = -a_f \dot{\phi}_f \sin\theta_f \tag{A.3d}$$

where $\dot{\phi}_f$ is the spin rate of the front wheel. These equations (and all others in this chapter unless otherwise indicated) assume the sign convention used in chapter III.

Equations (A.3a-d) represent four nonholonomic constraints imposed on the Basic bicycle model which has seven generalized coordinates: X_r, Y_r, θ_r, ψ, χ_r, ϕ_r, ϕ_f. Hence, for the given assumptions, the bicycle has three degrees of freedom.[6] However, as a consequence of linearizing the equations of motion, for the case of linearized equations of motion in the derivation it is shown that $\dot{Y}_r$ is constant to first order. Hence, for the linearized model, only two degrees of freedom exist.

What follows is the linearization and simplification of the nonlinear nonholonomic constraints. As will be shown, as a result of the linearization, two constraints become holonomic and two remain nonholonomic.

[6] See Neimark and Fufaev [1967] p. 335.

By assuming small angles of rotation the four nonlinear nonholomic constraints reduce to four linear nonholonomic constraints as follows,

$$\dot{Y}_r = a_r \dot{\phi}_r \tag{A.4a}$$

$$\dot{X}_r = -\dot{Y}_r \theta_r \tag{A.4b}$$

$$\dot{Y}_f = a_f \dot{\phi}_f \tag{A.4c}$$

$$\dot{X}_f = -\dot{Y}_f \theta_f \tag{A.4d}$$

Eliminating the auxiliary variables we can simplify these expressions.

Using equation (3.1b), (A.4c) becomes,

$$\dot{Y}_f = \dot{Y}_r = a_f \dot{\phi}_f \tag{A.5}$$

Taking the time derivative of equation $(A.1a)$ and substituting equation $(A.4b)$ for $\dot{X}_r$,

$$\dot{X}_f - \dot{X}_r = -c_w \dot{\theta}_r + c_f \dot{\psi} \tag{A.1a}$$

$$\dot{X}_r = \dot{X}_f + c_w \dot{\theta}_r - c_f \dot{\psi} = -\dot{Y}_r \theta_r \tag{A.6}$$

Substituting $(A.1d)$ into $(A.6)$ and cancelling the $\dot{X}_f$ and $-\dot{Y}_r \theta_r$ terms,

$$\theta_r = \theta_f - \psi \cos \lambda \tag{A.1d}$$

$$\dot{X}_f + c_w \dot{\theta}_r - c_f \dot{\psi} = -\dot{Y}_r(\theta_f - \psi \cos \lambda) \tag{A.7a}$$

$$c_w \dot{\theta}_r - c_f \dot{\psi} = \dot{Y}_r \psi \cos \lambda \tag{A.7b}$$

Rewriting equations $A.4a$, $A.4b$, $A.7b$, and $A.5$ we have the four linear constraints expressed in terms of the generalized coordinates,

$$\dot{Y}_r = a_r\dot{\phi}_r \tag{A.8a}$$

$$\dot{Y}_r = a_f\dot{\phi}_f \tag{A.8b}$$

$$\dot{\theta}_r = \frac{c_f}{c_w}\dot{\psi} + \psi\dot{Y}_r\frac{\cos\lambda}{c_w} \tag{A.8c}$$

$$\dot{X}_r = -\dot{Y}_r\theta_r \tag{A.8d}$$

As is shown in chapter III the time derivatives of these constraint equations is sometimes required. Taking their time derivatives we have,

$$\ddot{Y}_r = a_r\ddot{\phi}_r \tag{A.8e}$$

$$\ddot{Y}_r = a_f\ddot{\phi}_f \tag{A.8f}$$

$$\ddot{\theta}_r = \frac{c_f}{c_w}\ddot{\psi} + \dot{\psi}\dot{Y}_r\frac{\cos\lambda}{c_w} \tag{A.8g}$$

$$\ddot{X}_r = -\ddot{Y}_r\theta_r - \dot{Y}_r\dot{\theta}_r = -\ddot{Y}_r\theta_r - \frac{c_f}{c_w}\dot{Y}_r\dot{\psi} - \psi\dot{Y}_r^2\frac{\cos\lambda}{c_w} \tag{A.8h}$$

Equations (A.8a-h) are used to simplify Lagrange's equations by reducing the number of degrees of freedom in the final equations. And as is shown in chapter III $\ddot{Y}_r$ is zero to first order so we have eliminated this term in the expression presented in chapter III.

Note that the generalized coordinate χ_r is not present in these relations, and they are not dependent on the radii of the wheels.

Section 3: Derivation of the Kinetic Energy of the Rear Part of the Basic Bicycle Model to Second Order

Introduction

In Chapter III the contributing kinetic energy for the rear part of the Basic bicycle model to second order, KE_r^+ was stated without proof. The objective of this section is to show the derivation of KE_r^+ and give its theoretical justification. We also discuss how the noncontributing terms can be derived and why they can be eliminated. To accomplish the derivation the definition of kinetic energy for a rigid body is applied to the two rigid bodies that make up the rear part of the Basic bicycle model. In doing so we have simplified the algebra by noting how the rear wheel interacts with the rear frame.

The notation used here is identical to that presented in Chapter III. We note however, that the upper subscript for the kinetic energy terms, KE^*, indicates what type kinetic energy the terms from (rotation, translational, etc.), and the lower subscript, KE_*, designates what component or part of the bicycle the term is representative of. Vector quantities are indicated by an arrow overhead, $\vec{*}$; tensor quantities are in **bold face**; and the overhead bar, $\bar{*}$, indicates center of mass.

Definition of Kinetic Energy for a Rigid Body

By definition, the translational and rotational kinetic energy of a rigid body can be represented as,[7]

$$KE^{trans} = \frac{1}{2}m\left(\vec{V}\cdot\vec{V}\right) \tag{A.9}$$

$$KE^{rot} = \frac{1}{2}\vec{\omega}\cdot\overline{\mathrm{B}}\cdot\vec{\omega} \tag{A.10}$$

where,

KE^{trans}	-	the translational kinetic energy.
KE^{rot}	-	the rotational kinetic energy.
m	-	the total mass of the rigid body.
$\vec{V}$	-	the velocity vector of the center of mass relative to the inertial reference XYZ.
$\overline{\mathrm{B}}$	-	is the inertia tensor of the rigid body for the point at the center of mass.
$\vec{\omega}$	-	the angular velocity vector of the body relative to the inertial reference XYZ.

In order to evaluate equations $(A.9)$ and $(A.10)$ both the $\vec{\omega}$ and $\vec{V}$ must be measured with respect to the same inertial reference frame and expressed in the same coordinate system. The quantities $\vec{\omega}$ and $\overline{\mathrm{B}}$ also must be expressed, but not necessarily measured, from the the same reference frame. In this derivation we are using the inertial reference XYZ to measure $\vec{\omega}$ and $\vec{V}$, and expressing all quantities in the body fixed $\overline{x}_r\overline{y}_r\overline{z}_r$ axes with origin at $\overline{P}_r$ shown in figure A.6.

The rear part of the Basic bicycle model is composed of two rigid bodies: the rear frame with rigidly attached rider and the rear wheel as shown in figure A.9. Equations (A.9) and (A.10) apply to *each* component of the rear part. The total

[7] See Meirovitch [1983] pp. 132–133.

kinetic energy of the rear part of the Basic bicycle model is the sum of the total translational kinetic energy and rotational kinetic energy due to each component.

For the case of the Basic bicycle model, certain simplifications can be made in applying the equations (A.9) and (A.10). In calculating the translational kinetic energy of the rear part, KE_r^{trans}, we note that the rear wheel is constrained to only rotate relative to the bicycle. Because of this we can treat the rear part *translational* kinetic energy, KE_{trans}, as if the rear frame and rear wheel are one rigid body. Similarly, we can calculate the rotational kinetic energy of the rear part, KE_r^{rot}, by treating the rear wheel as if it were rigidly attached to the rear frame, and then *add* the additional rotational kinetic energy due to the spinning of the rear wheel relative to the rear frame, KE_{rwheel}^{spin}.

The total kinetic energy of the rear part of the Basic bicycle model, KE_r, is then the sum of the rear translational and rear rotational kinetic energies as if the rear wheel was welded to the rear frame, *plus* the additional rotational kinetic energy due to the rear wheel spinning. Mathematically this is expressed as follows,[8]

$$KE_r = KE_r^{trans} + KE_r^{rot} + KE_{rwheel}^{spin} \tag{A.11}$$

We can further define each term of the rear kinetic energy of the Basic bicycle. For the translational term,

$$KE_r^{tran} = \frac{1}{2} m_r \overline{V}^2 \tag{A.12}$$

[8] Note we have left out the + superscript in the KE_r equation (A.11) because in general the equation is true. However, as we will discuss next carrying all terms of the KE becomes messy and many terms that do not contribute can be neglected.

where $\vec{\overline{V}}$ is the velocity of the center of mass of the rear part (rear frame + rider + rear wheel); m_r is the mass of the rear frame, rider, and rear wheel. And for the rotational term,

$$KE_r^{rot} = \frac{1}{2}\vec{\omega}_r \cdot \overline{\mathbf{R}} \cdot \vec{\omega}_r \tag{A.13}$$

where $\overline{\mathbf{R}}$ is the inertia tensor of the rear part about the $\overline{x}_r\overline{y}_r\overline{z}_r$ axes; $\vec{\omega}$ the angular velocity of the rear part expressed in the body fixed axes $\overline{x}_r\overline{y}_r\overline{z}_r$ with origin at $\overline{P}_r$.

KE_{rwheel}^{spin} is the additional rotational kinetic energy due to the rotation of the rear wheel relative to the rear frame, the 'spin'. The theoretical value of KE_{rwheel}^{spin} is not as easily expressed without further discussion, so we won't express it mathematically yet.[9]

Next we will form the definition of KE_r^{trans}, KE_r^{rot}, and KE_{rwheel}^{spin} in terms of the bicycle design parameters.

Translational Kinetic Energy of the Rear Part

To calculate KE_r^{trans} we need $\overline{V}^2$, which can be expressed in terms of the cartesian coordinates $\overline{X}_r$, $\overline{Y}_r$, and $\overline{Z}_r$, which locates the rear center of mass $\overline{P}_r$ as,

$$\overline{V}_r^2 = \dot{\overline{X}}_r^2 + \dot{\overline{Y}}_r^2 + \dot{\overline{Z}}_r^2 \tag{A.14}$$

We can express $\overline{X}_r$, $\overline{Y}_r$, and $\overline{Z}_r$ in terms of the generalized coordinates X_r, Y_r, θ_r, and χ_r to second order as,

$$\overline{X}_r = X_r - \overline{l}_r\theta_r + \overline{h}_r\chi_r \tag{A.15a}$$

[9] If we wished to add additional mass such as saddlebags, passengers, etc... we could do so here provided we maintained symmetry about the plane of the bicycle when in its vertical position.

$$\overline{Y}_r = Y_r + \overline{l}_r(1 - \frac{\theta_r^2}{2}) + \overline{h}_r\theta_r\chi_r + \left(\frac{\psi^2}{2}\sin\lambda - \chi_r\psi\right)\frac{c_f}{c_w}(\overline{h}_r - a_r) \qquad (A.15b)$$

$$\overline{Z}_r = \overline{h}_r(1 - \frac{\chi_r^2}{2}) \qquad (A.15c)$$

where, as shown in Chapter III, $\overline{l}_r$ and $\overline{h}_r$ locate $\overline{P}_r$ in the $x_r y_r z_r$ coordinates axes.

Note here that we have approximated $\overline{X}_r$, $\overline{Y}_r$, and $\overline{Z}_r$ to second order. In theory we could have solved for the nonlinear expressions for these variables but this would incorporate solving for $\overline{h}_r$ and $\overline{l}_r$ as functions of the steer angle ψ. This is a tedious task and, as far as we know, as not been solved in closed form. This is mainly because the pitching motion of the bicycle is a function of ψ^4, so that the closed form solution is one which requires the roots to a fourth order polynomial, thus making exact nonlinear equations lengthly and cumbersome.

Rewriting $\overline{V}_r^2$ to second order and keeping only contributing terms,

$$\overline{V}_r^2 = \dot{Y}_r^2 + (\dot{X}_r - \overline{l}_r\dot{\theta}_r + \overline{h}_r\dot{\chi}_r)^2 \qquad (A.14a)$$

And the contributing translational kinetic energy for the rear, KE_r^{trans}, becomes,

$$KE_r^{trans} = \frac{1}{2}m_r\left(\dot{Y}_r^2 + \left(\dot{X}_r - \overline{l}_r\dot{\theta}_r + \overline{h}_r\dot{\chi}_r\right)^2\right) \qquad (A.12a)$$

where m_r includes the mass of the rear wheel so the translational kinetic energy of the rear wheel is included in KE_r^{trans}.

Rotational Kinetic Energy of the Rear Part[10]

To calculate KE_r^{rot} we need the angular velocity of the rear part $\vec{\omega}_r$. This can be found by projecting the components of $\dot{\chi}_r$ and $\dot{\theta}_r$ on to the rear body fixed axes $\overline{x}_r\overline{y}_r\overline{z}_r$. It can be shown that the angular velocity due to pitch of the rear frame is of second order and because $\vec{\omega}$ is squared pitching effects can be neglected in the rotation kinetic energy formulation.[11] Therefore the angular velocity of the rear frame for the Basic bicycle model is due to yawing, $\dot{\theta}_r$, and leaning, $\dot{\chi}_r$ rotations. Expressing it in the rear body fixed coordinates we have,

$$\vec{\omega}_r = -\dot{\theta}_r \sin\chi_r \vec{\mathbf{i}}_r + \dot{\chi}_r \vec{\mathbf{j}}_r + \dot{\theta}_r \cos\chi_r \vec{\mathbf{k}}_r \qquad (A.16)$$

Because the equilibrium position for the variables χ_r and θ_r is zero, there are no zero order terms in this expression we can linearize to first order as follows,

$$\vec{\omega}_r \simeq \dot{\chi}_r \vec{\mathbf{j}}_r + \dot{\theta}_r \vec{\mathbf{k}}_r \qquad (A.16a)$$

Calculating the rear part rotational kinetic energy as if the rear where glued to the rear frame and assuming symmetry about the plane of the rear frame we can add up the rotation kinetic energy due to the rear frame and rear wheel. KE_r^{rot} is then,

$$KE_r^{rot} = \frac{1}{2}\begin{pmatrix} 0 \\ \dot{\chi}_r \\ \dot{\theta}_r \end{pmatrix} \cdot \begin{pmatrix} R_{\overline{x}\overline{x}} & 0 & 0 \\ 0 & R_{\overline{y}\overline{y}} & R_{\overline{y}\overline{z}} \\ 0 & R_{\overline{z}\overline{y}} & R_{\overline{z}\overline{z}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \dot{\chi}_r \\ \dot{\theta}_r \end{pmatrix}_{\overline{x}_r,\overline{y}_r,\overline{z}_r} \qquad (A.13a)$$

[10] Remember KE_r^{rot} is the kinetic energy of the rear part (rear frame + rear wheel + rider) as if the rear wheel were glued to the rear frame. It does not include the kinetic energy of the rotation of the rear wheel relative to the rear frame. We designate this KE_{rwheel}^{spin}.

[11] Pitching effects are not negligible in the potential energy expression.

Expanding $(A.15a)$ we have,

$$KE_r^{rot} = \frac{1}{2}\left(R_{\overline{yy}}\dot{\chi}_r^2 + 2R_{\overline{yz}}\dot{\chi}_r\dot{\theta}_r + R_{\overline{zz}}\dot{\theta}_r^2\right) \qquad (A.13b)$$

where $\overline{\mathbf{R}}$, the inertia tensor of the rear part, *includes the rear wheel* as if it was glued to the rear frame. This is o.k. because we have not put in any constraints yet.

Kinetic Energy Due Spin of Rear Wheel

The rotation of the rear wheel relative to the rear frame is called the spin of the rear wheel. In calculating the additional rotational kinetic energy due to the spinning of the rear wheel relative to the rear frame, we note that the angular velocity of the rear wheel, $\vec{\omega}_{rwheel}$ will be the same as that of the rear frame *plus* the additional angular velocity due to its motion relative to the rear frame, that is its 'spin'.

Therefore, the angular velocity of the rear wheel is the angular velocity of the rear frame, $\vec{\omega}_r$, *plus* an additional $\dot{\phi}_r$ due to the rear wheel spinning in the body fixed $\overline{x}_r$ direction. Linearizing this as we did the center of mass velocity $\overline{V}_r$ and keeping terms to second order in the $\vec{\mathbf{i}}$ direction because the $\dot{\phi}_r$ equilibrium position is not zero we have,

$$\vec{\omega}_{rwheel} = \left(-\dot{\phi}_r - \dot{\theta}_r\chi_r\right)\vec{\mathbf{i}}_r + \dot{\chi}_r\vec{\mathbf{j}}_r + \dot{\theta}_r\vec{\mathbf{k}}_r \qquad (A.17)$$

Solving for KE_{rwheel}^{rot} we have,

$$KE_{rwheel}^{rot} = \frac{1}{2}\begin{pmatrix} -\dot{\phi}_r - \dot{\theta}_r\chi_r \\ \dot{\chi}_r \\ \dot{\theta}_r \end{pmatrix} \cdot \begin{pmatrix} C_r & 0 & 0 \\ 0 & A_r & 0 \\ 0 & 0 & A_r \end{pmatrix} \cdot \begin{pmatrix} -\dot{\phi}_r - \dot{\theta}_r\chi_r \\ \dot{\chi}_r \\ \dot{\theta}_r \end{pmatrix} \qquad (A.18a)$$

expanding,

$$KE^{rot}_{rwheel} = \overbrace{\frac{1}{2}C_r\dot{\phi}_r^2 + C_r\dot{\phi}_r\chi_r\dot{\theta}_r}^{KE^{spin}_{rwheel}} + \overbrace{\frac{1}{2}\left(A_r\dot{\chi}_r^2 + A_r\dot{\theta}_r^2\right)}^{\text{included in } KE^{rot}_r} \qquad (A.18b)$$

And as stated before $\overline{\mathrm{R}}$ includes the inertia of the rear wheel and thus the second group of terms in the above equation is included in KE^{rot}_r as indicated. KE^{spin}_{rwheel} is then just the first group of terms in $(A.18b)$ as indicated by the overbraces.

The total kinetic energy of the rear of part of the bicycle to second order is then equations (A.12a), (A.13b) and part of (A.18b) added,

$$\begin{aligned} KE_r = & \overbrace{\frac{1}{2}m_r(\dot{X}_r - \bar{l}_r\dot{\theta}_r + \bar{h}_r\dot{\chi}_r)^2 + \frac{1}{2}m_r\dot{Y}_r^2}^{KE^{trans}_r} \\ & + \overbrace{\frac{1}{2}(R_{\overline{yy}}\dot{\chi}_r^2 + 2R_{\overline{yz}}\dot{\chi}_r\dot{\theta}_r + R_{\overline{zz}}\dot{\theta}_r^2)}^{KE^{rot}_r} + \overbrace{C_r\dot{\phi}_r\chi_r\dot{\theta}_r + \frac{1}{2}C_r\dot{\phi}_r^2}^{KE^{spin}_{rwheel}} \qquad (A.19) \end{aligned}$$

where $(\ )^+$ indicates 'contributing parts' and,

m_r	-	mass of the rear part of the Basic bicycle(frame+rider+wheel).
$\bar{l}_r, \bar{h}_r$	-	length and height to the rear center of mass measured in $x_r y_r z_r$.
$R_{\overline{yy}}$, $R_{\overline{yz}}$, $R_{\overline{zz}}$	-	components of the rear part inertia tensor measured in $\bar{x}_r\bar{y}_r\bar{z}_r$.
C_r	-	polar mass moment of inertia of the rear wheel.
$\dot{\phi}_r$	-	spin rate of the rear wheel.
$\dot{Y}_r$	-	component of the velocity of the rear contact point in the Y_r direction

Also note the definition of $R_{\overline{y}\overline{z}}$ and $R_{\overline{z}\overline{y}}$,

$$R_{\overline{y}\overline{z}} = R_{\overline{z}\overline{y}} = -\int \overline{y}\,\overline{z}\,dm$$

Equation (A.19) is equivalent to kinetic energy of the rear part presented in Chapter III,

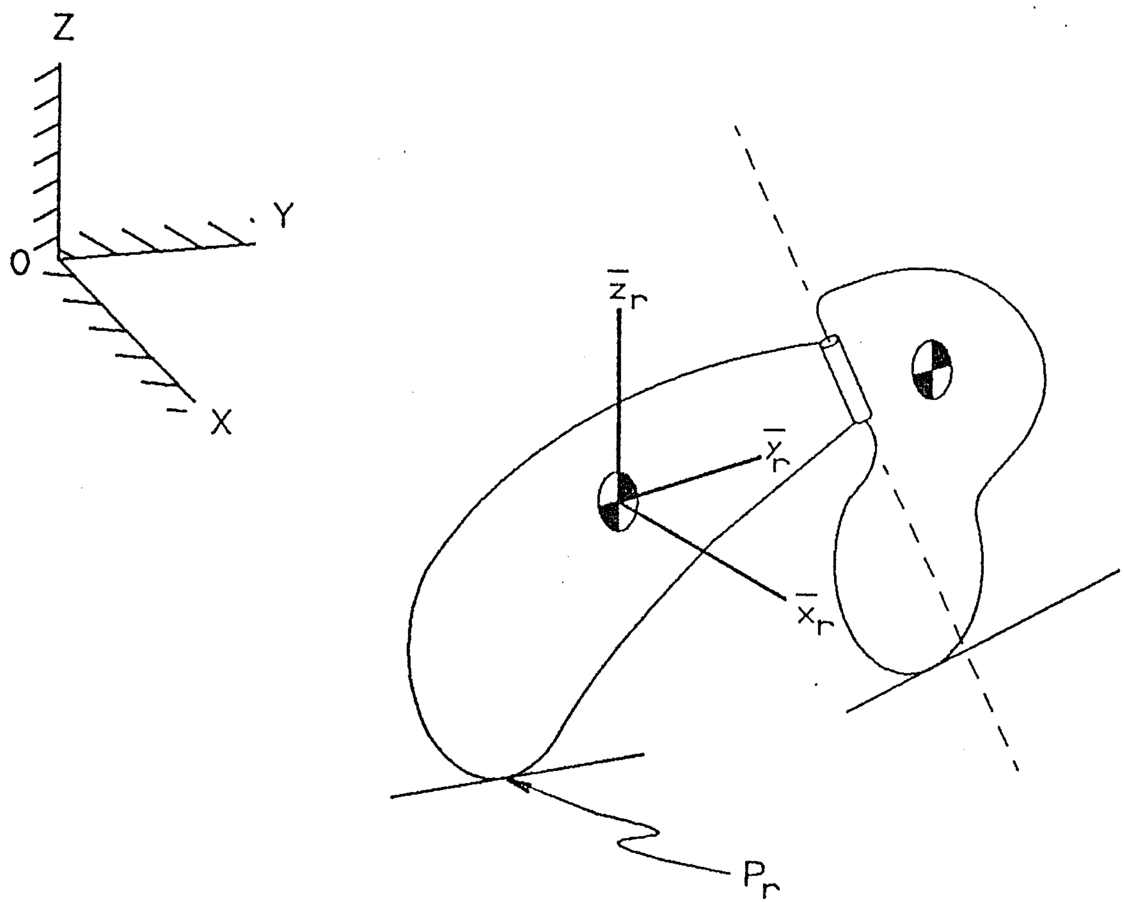

FIGURE A.6

Body fixed moving reference frames $\overline{x_r}\,\overline{y_r}\,\overline{z_r}$ at point $\overline{Pr}$.

Section 4: Derivation of the Potential Energy of the Basic Bicycle Model to Second Order

In Chapter III the approximate potential energy for the Basic bicycle model, expression $(A.20a)$, was given without proof. This section shows how the potential energy of the Basic bicycle model is derived to second order. To do this we will use small angle assumptions, just as we did in deriving the relation between the generalized coordinates and auxiliary variables.

There is, of course, the possibility of deriving nonlinear equations and then linearizing about the vertical equilibrium position for small disturbances. However, this method results in exceedingly complicated geometric relations which are more clearly understood using small angle approximations. Our aim is therefore to quickly, sensibly, give a method for seeing where the potential energy term comes from without going into the mathematical clutter. It is felt by the author that this is the quickest and easiest method to understanding the equations and arrive at the correct expression for the potential energy term to second order.

Using the ground as a reference as mentioned in Chapter III, the change in potential energy can be expressed as,

$$PE_t = m_r g \overline{Z}_r + m_f g \overline{Z}_f \tag{A.20}$$

where $\overline{Z}_r$ and $\overline{Z}_f$ are the locations of the rear and front part center of mass, respectively, as shown in figure A.7. As we will show the quantities $\overline{Z}_r$ and $\overline{Z}_f$ are functions of both the lean angle of the rear frame, χ_r, and steer angle of the front frame relative to the rear frame, ψ.

The lean angle dependence is illustrated in figure A.8. When the bicycle with no steer angle is leaned over the center of mass is lowered just as with an inverted pendulum.

The steer angle effects the potential energy three ways. First, imagine the bicycle is put in the vertical equilibrium position and the rear frame *held fixed.* Assuming a geometry and mass configuration as shown in figure A.9. A rotation ψ about the steering axis would result in lowering the front center of mass, m_f.[12]

Second, with the frame still held fixed, this same ψ also *raises* the front wheel *off the ground* as shown in figure A.11. As a result, an overall drop of the center of mass of the total bicycle after the rear frame is freed and lowered (and therefore a decrease in potential energy). This can be thought of as a pitching of the rear frame and fork assembly about the rear contact point as shown in figure A.10.[13] (Remember, no constraints have been added to the problem yet.)

Unlike in the derivation of the relations between the auxiliary variables and the generalized coordinates, and kinetic energy expressions, the raising and lowering of the bicycle is of second order and must be included in potential energy derivation

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[12] In this deraviation we make some assumptions about the location of the center of mass. In general when checked the derived equations were found valid.

[13] This discussion assumes that the the bicycle has the typical design configuration of with positive head angle, positive trail, and front center of mass located in front of the steering axis. The final equations derived however are valid for design configurations with negative design parameters.

to derive the correct linearized equations of motion to first order. This is due to the fact that the terms representing the up and down motion (drop) in the potential energy are not squared, as was the case in the kinetic energy expressions.

The third effect due to the steer angle is that the center of mass of the front fork/handlebar assembly and front contact point do not remain in the plane of the rear frame as shown in figure A.11. As a result, steering the bicycle while the bicycle is already leaning at a particular angle could result in raising or lowering the center of mass of the total bicycle.

Adding all effects of lean angle and steer angle yields the correct expression for the potential energy to second order. This can be done in three steps; Starting from the equilibrium position and assuming positive small angle rotations of χ_r and ψ. Each step contributes to the change in potential energy.

1. Give a small positive lean to the bicycle from the equilibrium position with no steer angle and hold the frame at a constant χ_r. Find the change in height of the total center of mass, $\overline{P}_t$.
2. At the same constant lean angle, still holding the rear frame fixed, and give a small positive steer. Find change in height of the front center of mass, $\overline{P}_f$.[14]
3. Free the rear frame by allowing the rear frame to rotate in its plane about the rear contact point. Raise or lower the front contact point so that it is at the ground level. Find the change in height of the total center of mass due to this raising or lowering of the bicycle.

[14] Practically speaking the front wheel would dig into the ground. We will allow this for now and raise or lower the frame in step 3.

To calculate the total change in potential energy we subtract all changes in potential energy due to lowering the height of the center of mass(es), from the potential energy of the bicycle in its equilibrium position (or add potential if the height was increased due to ψ or χ_r). The results and figures describing the changes in potential energy to second order are as follows:

1) Change due to pure leaning.

$$-\frac{g}{2}\left(\chi_r^2\overline{h}_r m_r + \chi_r^2\overline{h}_f m_f\right) = -\frac{\chi_r^2}{2}g\overline{h}_t m_t$$

2) Change due to positive steer with rear held fixed and constant lean angle.

$$-\frac{g}{2}\left(m_f d\psi^2 \sin\lambda - 2m_f d\chi_r\psi\right)$$

3) Change due to freeing the rear frame and raising the front wheel contact to be at ground level.

$$-\frac{g\overline{l_t}}{2c_w}\left(m_t c_f\psi^2 \sin\lambda - 2m_t c_f\chi_r\psi\right)$$

Summing all the changes in potential energy to second order and adding them to the potential energy in the equilibrium position we get the the equation presented in Chapter III.

$$\begin{aligned} PE_t = & \overbrace{m_r g\overline{h}_r + m_f g\overline{h}_f}^{constant} - \frac{g}{2}m_t\overline{h}_t\chi_r^2 \\ & - \frac{gm_f d}{2}\left(\psi^2 \sin\lambda - 2\chi_r\psi\right) \\ & - \frac{g\overline{l}_t m_t c_f}{2c_w}\left(\psi^2 \sin\lambda - 2\chi_r\psi\right) \end{aligned} \qquad (A.20a)$$

where,

$$m_t = m_r + m_f$$

$$\bar{l}_t = \frac{m_r \bar{l}_r + m_f(c_w + \bar{l}_f)}{m_t}$$

$$\bar{h}_t = \frac{m_r \bar{h}_r + m_f \bar{h}_f}{m_t}$$

if we further let,

$$\nu = m_f d + \frac{c_f}{c_w} m_t \bar{l}_t$$

and disregard constant terms then we simply have,

$$PE_t = -\frac{g}{2}(m_t \bar{h}_t \chi_r^2 - 2\psi \chi_r \nu + \psi^2 \nu \sin \lambda) \tag{A.20b}$$

where $(\ )_t$ stands for total.

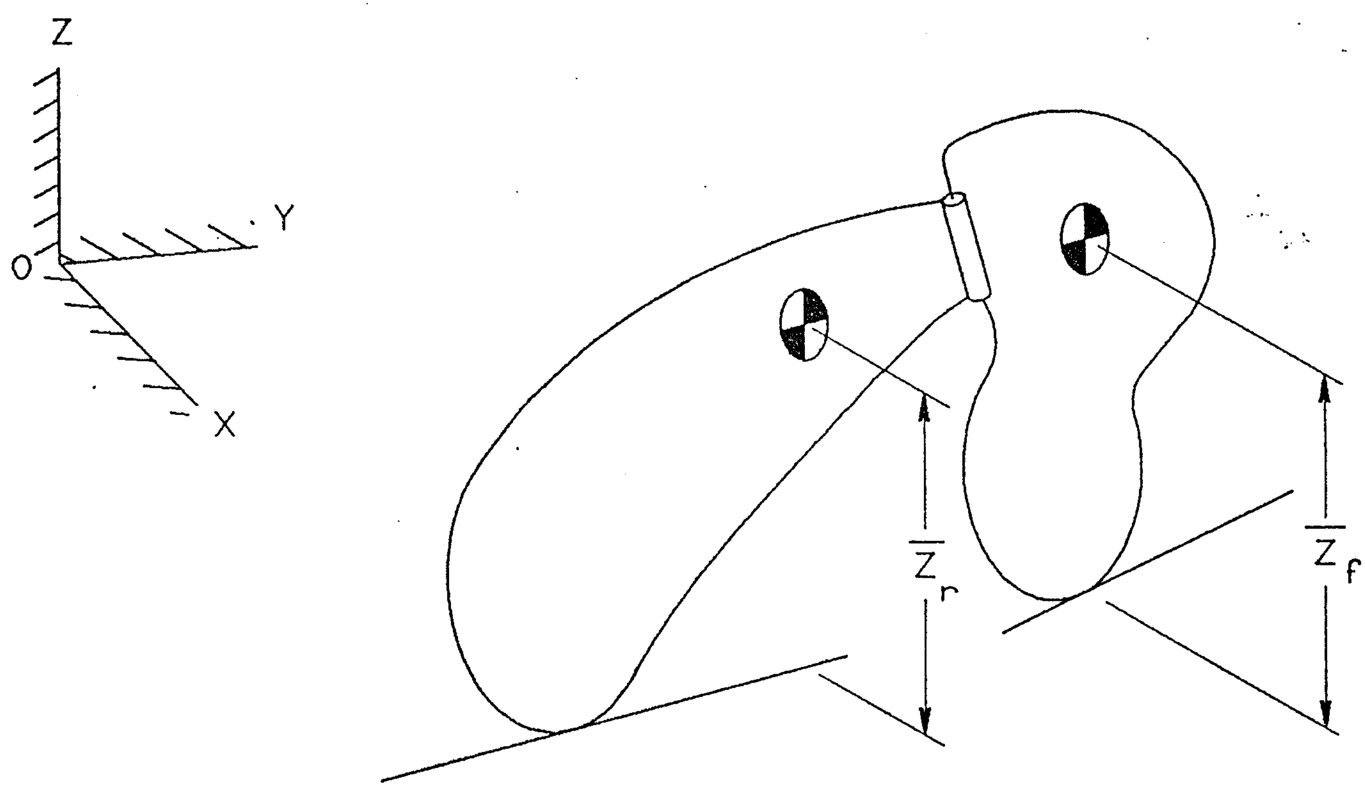


FIGURE A.7

Location of the rear and front center of mass, $\overline{Z}_r$ and $\overline{Z}_f$, respectively.

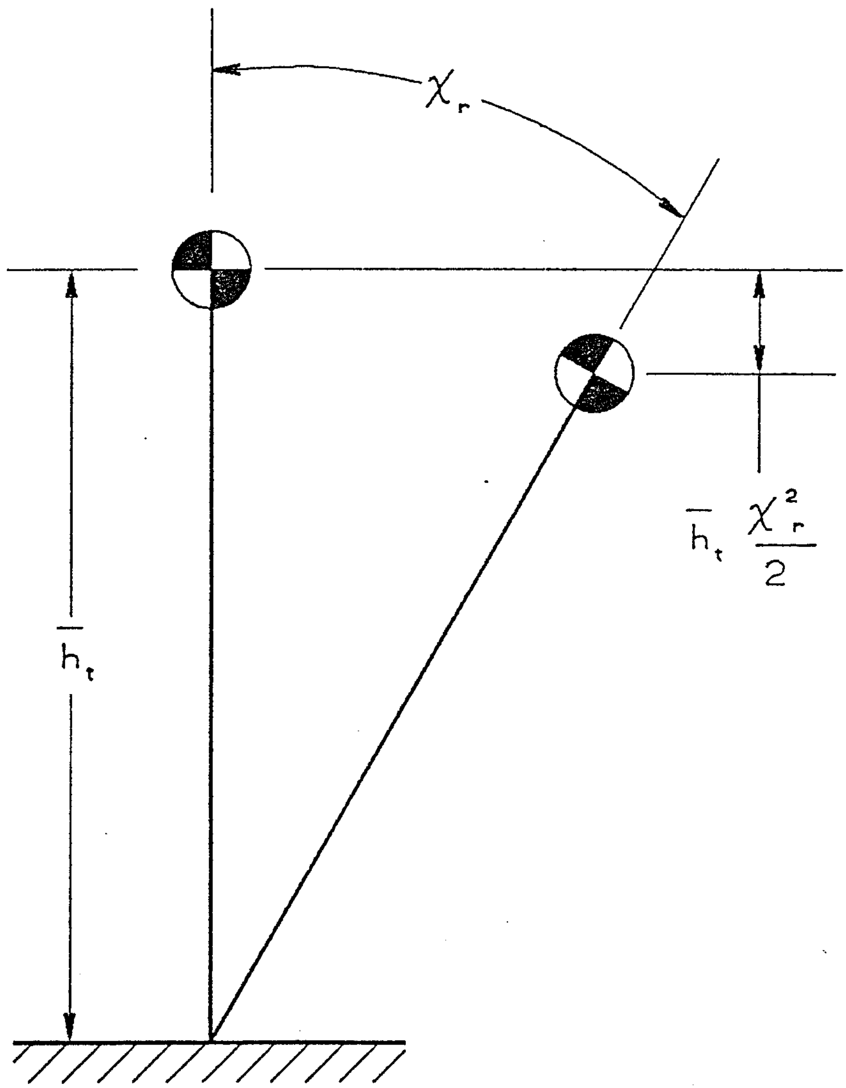


FIGURE A.8

Lowering of the center of mass of the bicycle due to leaning.

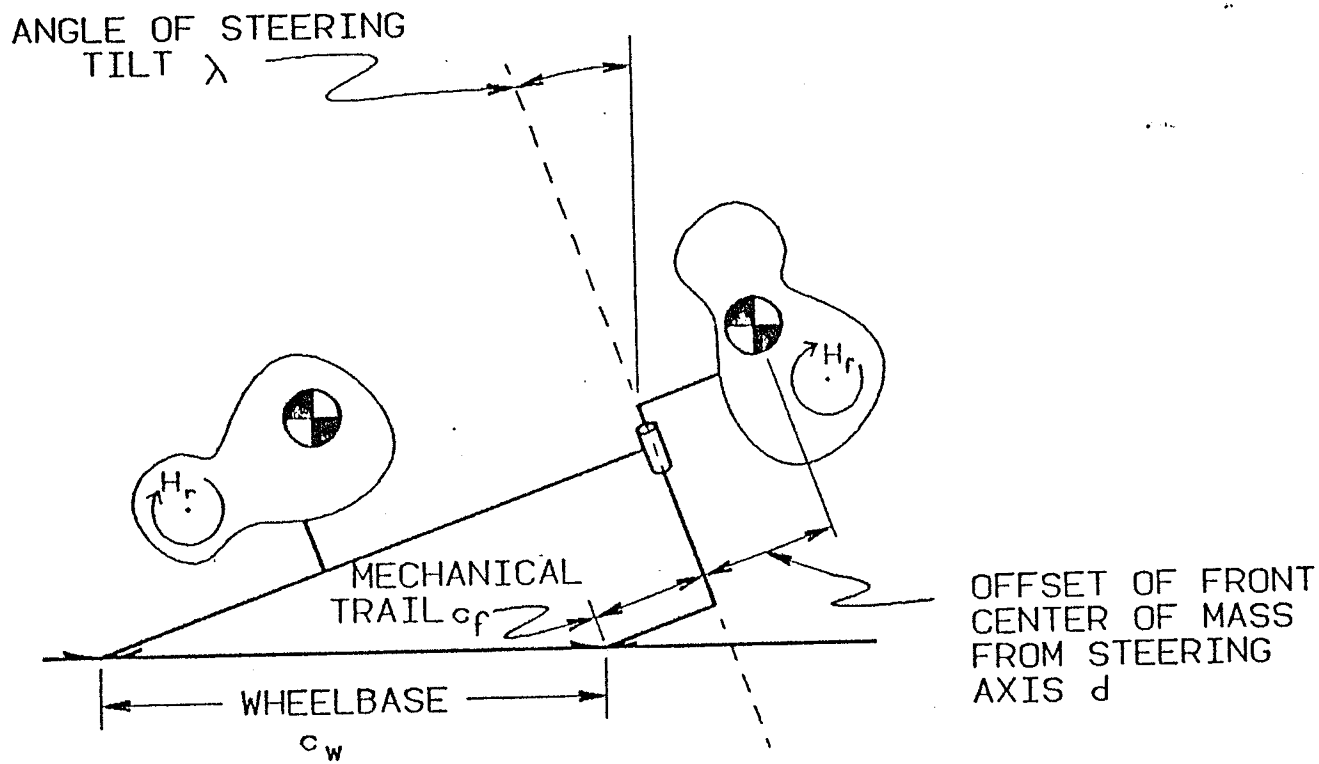


FIGURE A.9

Assumed mass and geometry configurations for potential energy derivation.

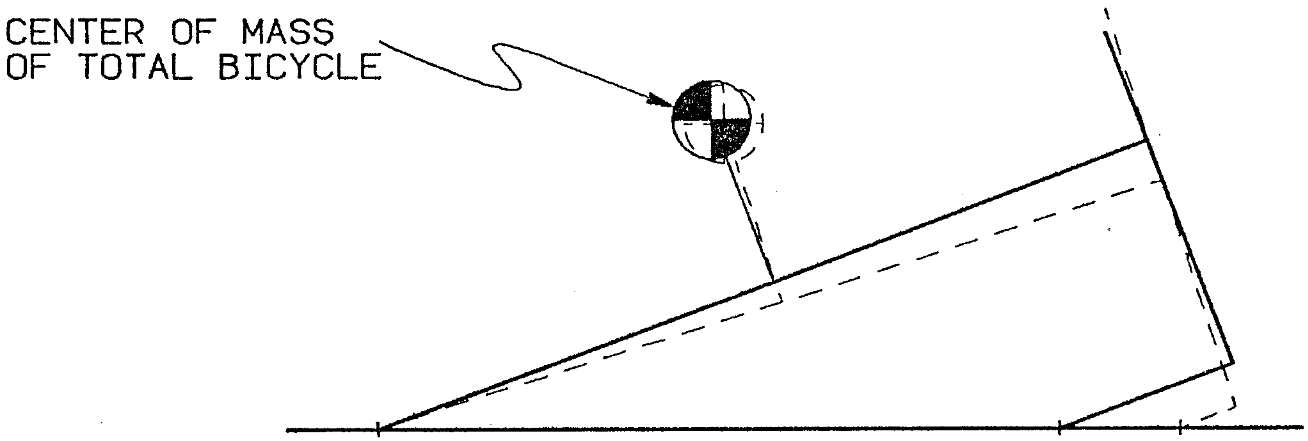


FIGURE A.10

Pitching of the Basic bicycle model about the rear contact point due to pure steering.

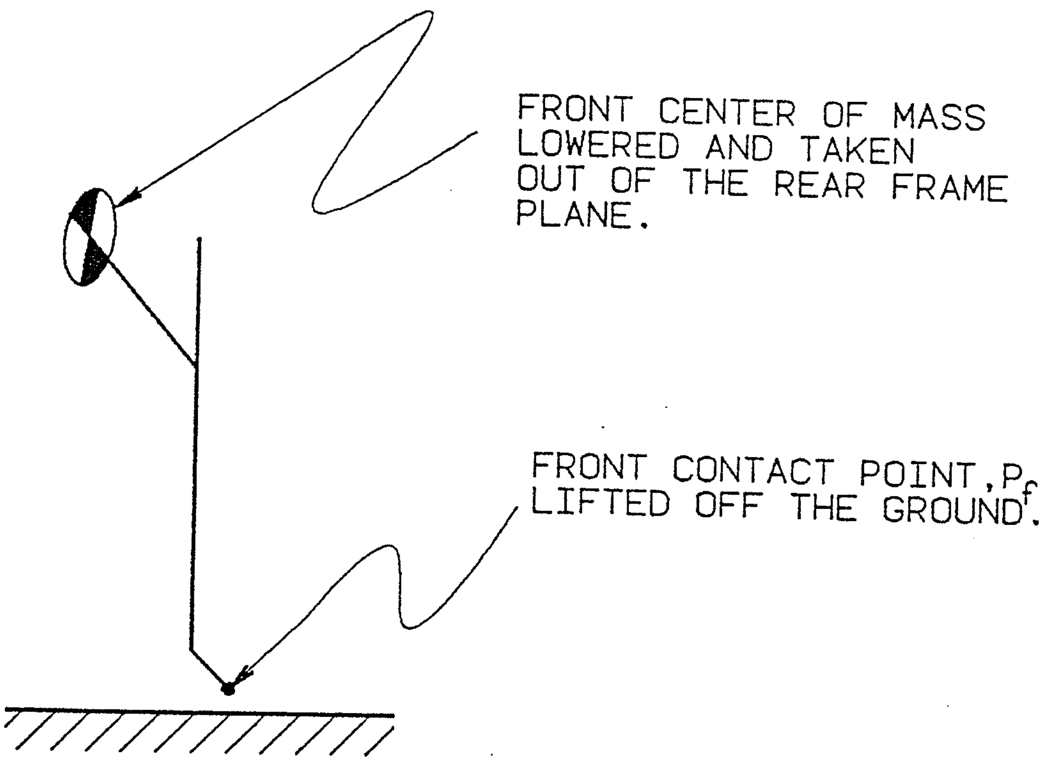
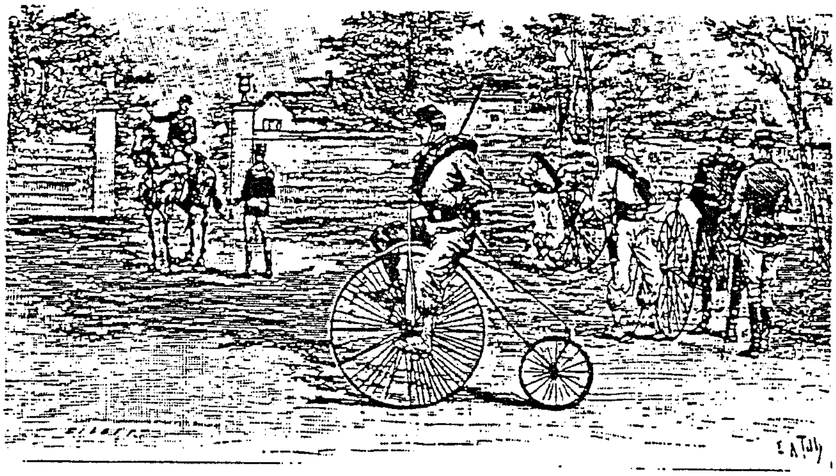


FIGURE A.11

Rear view of front assembly of Basic bicycle model with rear frame held fixed in the vertical equilibrium position and a positive steer angle, ψ.



APPENDIX B

PC-BASED COMPUTER PROGRAM FOR DESIGN INVESTIGATION

PROGRAM DESIGN

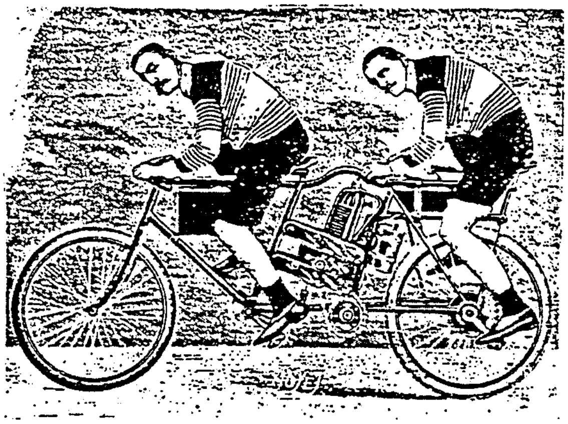
```
C       This program calculates the Routh Hurwitz Criteria equations
C       as functions of velocity. An option for displaying the
C       stable velocity regions is then given.

        REAL var(17)
        REAL mr,hr,lr,Ryy,Rzz,Ryz,cw,cf,lambda,Ryp,Rzp,alphar
        REAL mf,hf,lf,Fyy,Fzz,Fyz,Fyp,Fzp,alphaf,u
        REAL CrR,CfR,g
        REAL d,Tyy,Tzz,Tyz,Fll,Fly,Flz,Ktr,Khd,mt,ht,lt,nu
        REAL a1,a2,a3,a4,a5,a6
        REAL b1,b2,b3,b4,b5,b6,b7
        REAL a0,b0,c0,d0,e0,f0,g0,h0,i0,j0,k0,l0,m0
        INTEGER ANS
        INTEGER I,J,YES1

C       Intialize arrays to zero.

        CALL INIT(var)

C       Read and echo the baseline dataset.

        OPEN(7,IOSTAT=YES1,FILE='BICDES.DAT')

                READ(7,1000) (var(I),I=1,17)

1000    FORMAT  (T2,G17.4)

C       temporarily echo data
        Do 999 I=1,17
                Print *, var(I)
999     Continue

C       Intialize design data to variable names.

                mr      =       var(1)
                lr      =       var(2)
                hr      =       var(3)
                Ryp     =       var(4)
                Rzp     =       var(5)
                alphar  =       var(6) * 3.1415927/180.0
                cw      =       var(7)
                cf      =       var(8)
                lambda  =       var(9) * 3.1415927/180.0
                mf      =       var(10)
                d       =       var(11)
                u       =       var(12)
                Fyp     =       var(13)
                Fzp     =       var(14)
                alphaf  =       var(15)* 3.1415927/180.0
                CrR     =       var(16)
                CfR     =       var(17)
                g       =       386.4
```

```
C       Calculate front and rear inertias.

             Ryy      =  Rzp*(cos(alphar))**2+Ryp*(sin(alphar))**2
             Rzz      =  Rzp*(sin(alphar))**2+Ryp*(cos(alphar))**2
             Ryz      =  Rzp*cos(alphar)*sin(alphar)
     *                  -Ryp*sin(alphar)*cos(alphar)
             Fyy      =  Fzp*(cos(alphaf))**2+Fyp*(sin(alphaf))**2
             Fzz      =  Fzp*(sin(alphaf))**2+Fyp*(cos(alphaf))**2
             Fyz      =  Fzp*cos(alphaf)*sin(alphaf)
     *                  -Fyp*sin(alphaf)*cos(alphaf)

C       Calculate front design lengths

             hf       =       (cf+d)*sin(lambda)+u*cos(lambda)
             lf       =       (cf+d)*cos(lambda)-u*sin(lambda)

C       Calculate entire bicycle inertias and special front inertia
C       quantities.

             Tyy      =       mr*hr**2+Ryy+mf*hf**2+Fyy
             Tzz      =       mr*lr**2+Rzz+mf*(cw+lf)**2+Fzz
             Tyz      =      -mr*hr*lr+Ryz-mf*hf*(cw+lf)+Fyz
             Fll      =       mf*d**2+Fyy*(sin(lambda))**2
     *                       -Fyz*sin(2*lambda)+Fzz*(cos(lambda))**2
             Fly      =      -mf*hf*d-Fyy*sin(lambda)+Fyz*cos(lambda)
             Flz      =       mf*(cw+lf)*d-Fyz*sin(lambda)
     *                       +Fzz*cos(lambda)

C       Calculate design length ratios and simplifying definitions.

             Ktr      =       cf/cw
             Khd      =       cos(lambda)/cw
             mt       =       mr+mf
             ht       =       (mr*hr+mf*hf)/mt
             lt       =       (mr*lr+mf*(cw+lf))/mt
             nu       =       mf*d+mt*lt*Ktr

        PRINT *, Ryy,Rzz,Ryz
        PRINT *, Fyy,Fzz,Fyz
        PRINT *, Tyy,Tzz,Tyz
        PRINT *, Fll,Fly,Flz
        PRINT *, Ktr,Khd,nu
        PRINT *, mt,ht,lt

C       Calculate a1-a6, coefficients for the lean equation.

             a1       =       Tyy
             a2       =      -g*mt*ht
             a3       =       Fly+Ktr*Tyz
             a4       =      -(CfR*cos(lambda)+Ktr*(CfR+CrR))
     *                       +(Tyz*Khd-Ktr*mt*ht)
             a5       =       g*nu
             a6       =      -(CfR+CrR)*Khd-Khd*mt*ht

        Print *, a1,a2,a3,a4,a5,a6
```

```
C       Calculate b1-b7, coefficients for the steer equation.

                b1      =       Fll+2*Ktr*Flz+(Ktr**2)*Tzz
                b2      =       Khd*(Flz+Ktr*Tzz)+Ktr*nu
                b3      =      -g*nu*sin(lambda)
                b4      =       CfR*sin(lambda)*Khd+nu*Khd
                b5      =       Fly+Ktr*Tyz
                b6      =       CfR*cos(lambda)+Ktr*(CfR+CrR)
                b7      =       g*nu

        Print *, b1,b2,b3,b4,b5,b6,b7

C       Calculate a0-m0, coefficients for the fourth order polynomial.

                a0      =       a1*b1-a3*b5
                b0      =       a1*b2-a3*b6-a4*b5
                c0      =       a1*b3-a3*b7-b5*a5+b1*a2
                d0      =       a1*b4-b5*a6-b6*a4
                e0      =      -a4*b7-a5*b6+b2*a2
                f0      =      -b6*a6
                g0      =       a2*b3-a5*b7
                h0      =       a2*b4-a6*b7
                i0      =       b0*d0*f0-a0*f0**2
                j0      =       b0*c0*f0+b0*d0*e0-2*a0*e0*f0-h0*b0**2
                k0      =       b0*c0*e0-a0*e0**2-g0*b0**2
                l0      =       b0*d0-a0*f0
                m0      =       b0*c0-a0*e0

C       Print Routh Hurwitz criteria as function of velocity.

                PRINT *, 'A = ', a0 ,'  > 0'
                PRINT *, 'B = ', b0,'  V  > 0'
                PRINT *, 'C = ', c0,' + ',d0,'  V^2  > 0'
                PRINT *, 'D = ', e0,' V',' + ',f0,'  V^3  > 0'
                PRINT *, 'E = ', g0,' + ',h0,'  V^2  > 0'
                PRINT *, '6th= ',l0,' V^3 + ',m0,' V  > 0'
                PRINT *, '7th= ',i0,' V^4 + ',j0,' V^2 + ',k0,' > 0'


        WRITE(6,10) 'Want to know stable velocity regions? 1 yes 2 no  '
10      FORMAT(1X,A50)
        READ(6,20) ans
20      FORMAT(I1)

                        IF (ANS.EQ.1) THEN

                CALL PRTSTA(a0,b0,c0,d0,e0,f0,g0,h0,i0,j0,k0,l0,m0)

                        ENDIF

5000    STOP
        END
```

```
      SUBROUTINE PRTSTA(a0,b0,c0,d0,e0,f0,g0,h0,i0,j0,k0,l0,m0)
      REAL a0,b0,c0,d0,e0,f0,g0,h0,i0,j0,k0,l0,m0
      REAL VMPH,SPEED,VIPS
      REAL A,B,C,BIGD,E,ROUTH6,ROUTH7
      INTEGER J
      CHARACTER*1 PRT(7)


         DO 400 J=1,100

              SPEED    =       J
              VMPH     =       SPEED/2
              VIPS     =       VMPH*5280*12/3600
              A        =       a0
              B        =       b0*VIPS
              C        =       c0+d0*VIPS**2
              BIGD     =       e0*VIPS+f0*VIPS**3
              E        =       g0+h0*VIPS**2
              ROUTH6   =       l0*VIPS**3 + m0*VIPS
              ROUTH7   =       i0*VIPS**4 + j0*VIPS**2 + k0
              PRT(1)   =       ' '
              PRT(2)   =       ' '
              PRT(3)   =       ' '
              PRT(4)   =       ' '
              PRT(5)   =       ' '
              PRT(6)   =       ' '
              PRT(7)   =       ' '

              IF(A.LE.0) THEN

                      PRT(1)   =       'A'

              ENDIF

              IF(B.LE.0) THEN

                      PRT(2)   =       'B'

              ENDIF

              IF(C.LE.0) THEN

                      PRT(3)   =       'C'

              ENDIF

              IF(BIGD.LE.0) THEN

                      PRT(4)   =       'D'

              ENDIF
```

```
          IF(E.LE.0) THEN

                PRT(5)  =       'E'

          ENDIF

          IF(ROUTH6.LE.0) THEN

                PRT(6)  =       '6'

          ENDIF


          IF(ROUTH7.LE.0) THEN

                PRT(7)  =       '7'

          ENDIF

      WRITE(6,30) VMPH,PRT(1),PRT(2),PRT(3),PRT(4),PRT(5),PRT(6),
     *             PRT(7),ROUTH7
30    FORMAT(1X,'VEL=',T6,F6.2,T13,A1,TR2,A1,TR2,A1,TR2,A1,TR2,A1,
     *       TR2,A1,TR2,A1,G17.4)

400   CONTINUE

      END



      SUBROUTINE INIT(var)
      REAL var(17)
      INTEGER J

      DO 600 J=1,17

            var(J)          =       0.0

600   CONTINUE

      RETURN
      END


C>
```



APPENDIX C

COMPUTER PROGRAM VERIFICATION RUNS

Computer output for figure 5.5. The primitive bicycle model is never stable.

```
A =    6.03825024E+08   > 0
B =    2.05998160E+07   V   > 0
C =   -4.07941811E+09 +  244939.00000000   V^2   > 0
D =   -9.94980000E+07 V +   0.00000000E-01   V^3   > 0
E =    0.00000000E-01 +   0.00000000E-01   V^2   > 0
6th=    5.04569843E+12 V^3 +  -2.39558802E+16 V   > 0
7th=    0.00000000E-01 V^4 +  -5.02036904E+20 V^2 +    2.38356202E+24 > 0
Want to know stable velocity regions? 1 yes 2 no  1

VEL=  0.50      C  D  E  6          0.2345E+25
VEL=  1.00      C  D  E  6          0.2228E+25
VEL=  1.50      C  D  E  6          0.2034E+25
VEL=  2.00      C  D  E  6          0.1762E+25
VEL=  2.50      C  D  E  6          0.1412E+25
VEL=  3.00      C  D  E  6          0.9840E+24
VEL=  3.50      C  D  E  6          0.4786E+24
VEL=  4.00      C  D  E     7      -0.1046E+24
VEL=  4.50      C  D  E     7      -0.7655E+24
VEL=  5.00      C  D  E     7      -0.1504E+25
VEL=  5.50      C  D  E     7      -0.2321E+25
VEL=  6.00      C  D  E     7      -0.3215E+25
VEL=  6.50      C  D  E     7      -0.4187E+25
VEL=  7.00      C  D  E     7      -0.5236E+25
VEL=  7.50         D  E     7      -0.6364E+25
VEL=  8.00         D  E     7      -0.7569E+25
VEL=  8.50         D  E     7      -0.8852E+25
VEL=  9.00         D  E     7      -0.1021E+26
VEL=  9.50         D  E     7      -0.1165E+26
VEL= 10.00         D  E     7      -0.1317E+26
VEL= 10.50         D  E     7      -0.1476E+26
VEL= 11.00         D  E
```

```
 200.0                             mr, lbm                  1
 20.0                              lr, in                   2
 50.0                              hr, in                   3
 100000.0                          Ryp, lbm in^2            4
 20000.0                           Rzp, lbm in^2            5
 60.0                              alphar, degrees          6
 41.0                              cw, in                   7
 0.0                               cf, in                   8
 0.0                               lambda, degrees          9
 10.0                              mf, lbm                  10
 0.0                               d, in                    11
 30.00                             u, in                    12
 50.0                              Fyp, lbm in^2            13
 2000.0                            Fzp, lbm in^2            14
 45.00                             alphaf, degrees          15
 0.0                               CrR, lbm in              16
 0.0                               CfR, lbm in              17
```

Computer output for figure 5.6. The primitive bicycle model with wheels is never stable.

```
A =    6.03825024E+08   > 0
B =    2.05998160E+07   V   > 0
C =  -4.07941811E+09 +  549775.56250000   V^2   > 0
D =  -9.94980000E+07 V +    12682.92578125   V^3   > 0
E =    0.00000000E-01 +    0.00000000E-01   V^2   > 0
6th=    3.66700711E+12 V^3 +   -2.39558802E+16 V   > 0
7th=    4.65083798E+16 V^4 +   -6.68690532E+20 V^2 +    2.38356202E+24 > 0
Want to know stable velocity regions? 1 yes 2 no   1

VEL=  0.50      C  D  E  6        0.2332E+25
VEL=  1.00      C  D  E  6        0.2181E+25
VEL=  1.50      C  D  E  6        0.1940E+25
VEL=  2.00      C  D  E  6        0.1626E+25
VEL=  2.50      C  D  E  6        0.1263E+25
VEL=  3.00      C  D  E  6        0.8808E+24
VEL=  3.50      C  D  E  6        0.5158E+24
VEL=  4.00      C  D  E  6        0.2118E+24
VEL=  4.50      C  D  E  6        0.1903E+23
VEL=  5.00         D  E     7    -0.5692E+22
VEL=  5.50            E           0.2013E+24
VEL=  6.00            E           0.7102E+24
VEL=  6.50            E           0.1598E+25
VEL=  7.00            E           0.2949E+25
VEL=  7.50            E           0.4852E+25
VEL=  8.00            E           0.7406E+25
VEL=  8.50            E           0.1071E+26
VEL=  9.00            E           0.1488E+26
VEL=  9.50            E           0.2004E+26
VEL= 10.00            E           0.2630E+26
```

```
200.0              mr, lbm                1
20.0               lr, in                 2
50.0               hr, in                 3
100000.0           Ryp, lbm in^2          4
20000.0            Rzp, lbm in^2          5
60.0               alphar, degrees        6
41.0               cw, in                 7
0.0                cf, in                 8
0.0                lambda, degrees        9
10.0               mf, lbm                10
0.0                d, in                  11
30.00              u, in                  12
50.0               Fyp, lbm in^2          13
2000.0             Fzp, lbm in^2          14
45.00              alphaf, degrees        15
50.0               CrR, lbm in            16
50.0               CfR, lbm in            17
```

Computer output for figure 5.8. The model is self-stable after 6.5 mph.

```
A =    6.43872320E+08   > 0
B =    2.12858400E+07   V   > 0
C =   -4.35002522E+09 +   533447.18750000   V^2   > 0
D =   -1.02676024E+08 V +   12667.25781250   V^3   > 0
E =    0.00000000E-01 +  169280.62500000   V^2   > 0
6th=   3.19877441E+12 V^3 +  -2.64836875E+16 V   > 0
7th=   4.05197050E+16 V^4 +  -7.40612033E+20 V^2 +   2.71923973E+24 > 0
Want to know stable velocity regions? 1 yes 2 no  1

VEL=  0.50      C  D   6        0.2662E+25
VEL=  1.00      C  D   6        0.2494E+25
VEL=  1.50      C  D   6        0.2223E+25
VEL=  2.00      C  D   6        0.1864E+25
VEL=  2.50      C  D   6        0.1437E+25
VEL=  3.00      C  D   6        0.9695E+24
VEL=  3.50      C  D   6        0.4924E+24
VEL=  4.00      C  D   6        0.4395E+23
VEL=  4.50      C  D   6  7    -0.3321E+24
VEL=  5.00      C  D   6  7    -0.5861E+24
VEL=  5.50                7    -0.6628E+24
VEL=  6.00                7    -0.5009E+24
VEL=  6.50                7    -0.3324E+23
VEL=  7.00                      0.8129E+24
VEL=  7.50                      0.2116E+25
VEL=  8.00                      0.3962E+25
VEL=  8.50                      0.6439E+25
VEL=  9.00                      0.9645E+25
VEL=  9.50                      0.1368E+26
VEL= 10.00                      0.1866E+26
VEL= 10.50
```

```
200.0                 mr, lbm               1
20.0                  lr, in                2
50.0                  hr, in                3
100000.0              Ryp, lbm in^2         4
20000.0               Rzp, lbm in^2         5
60.0                  alphar, degrees       6
41.0                  cw, in                7
0.0                   cf, in                8
-2.0                  lambda, degrees       9
10.0                  mf, lbm               10
0.0                   d, in                 11
30.00                 u, in                 12
50.0                  Fyp, lbm in^2         13
2000.0                Fzp, lbm in^2         14
45.00                 alphaf, degrees       15
50.0                  CrR, lbm in           16
50.0                  CfR, lbm in           17
```

Computer output for figure 5.9. The model is self-stable after 7.0 mph.

```
A =    6.57631936E+08   > 0
B =    3.34873640E+07   V   > 0
C =  -4.18753459E+09 +  606022.18750000  V^2  > 0
D =  -1.77594864E+08 V +   13301.60546875  V^3  > 0
E =  -1.73520602E+09 +  101599.44531250  V^2  > 0
6th=   1.15465246E+13 V^3 +  -2.34374433E+16 V  > 0
7th=   1.53587326E+17 V^4 +  -2.47629300E+21 V^2 +   6.10823578E+24 > 0
Want to know stable velocity regions? 1 yes 2 no  1

VEL=  0.50       C  D  E  6           0.5917E+25
VEL=  1.00       C  D  E  6           0.5356E+25
VEL=  1.50       C  D  E  6           0.4457E+25
VEL=  2.00       C  D  E  6           0.3276E+25
VEL=  2.50       C  D  E  6           0.1890E+25
VEL=  3.00       C  D  E              0.3984E+24
VEL=  3.50       C  D  E     7       -0.1077E+25
VEL=  4.00       C  D  E     7       -0.2392E+25
VEL=  4.50       C  D  E     7       -0.3382E+25
VEL=  5.00          D  E     7       -0.3858E+25
VEL=  5.50          D  E     7       -0.3610E+25
VEL=  6.00          D  E     7       -0.2407E+25
VEL=  6.50          D  E              0.6381E+22
VEL=  7.00             E              0.3906E+25
VEL=  7.50                            0.9590E+25
VEL=  8.00                            0.1738E+26
VEL=  8.50                            0.2762E+26
VEL=  9.00                            0.4067E+26
VEL=  9.50                            0.5691E+26
VEL= 10.00                            0.7677E+26
VEL= 10.50
```

```
200.0              mr, lbm                1
20.0               lr, in                 2
50.0               hr, in                 3
100000.0           Ryp, lbm in^2          4
20000.0            Rzp, lbm in^2          5
60.0               alphar, degrees        6
41.0               cw, in                 7
1.0                cf, in                 8
0.0                lambda, degrees        9
10.0               mf, lbm               10
0.0                d, in                 11
30.00              u, in                 12
50.0               Fyp, lbm in^2         13
2000.0             Fzp, lbm in^2         14
45.00              alphaf, degrees       15
50.0               CrR, lbm in           16
50.0               CfR, lbm in           17
```

Computer output for figure 5.10. The model is self-stable after 5.5 mph.

```
A =    6.28236032E+08  > 0
B =    2.93819360E+07  V  > 0
C =  -4.24441088E+09 +  686129.25000000  V^2  > 0
D =  -1.36504560E+08 V +   12682.92578125  V^3  > 0
E =  -5.97219840E+07 +   18848.75195312  V^2  > 0
6th=   1.21919337E+13 V^3 +  -3.89519217E+16 V  > 0
7th=   1.54629405E+17 V^4 +  -2.17455098E+21 V^2 +   5.36867295E+24 > 0
Want to know stable velocity regions? 1 yes 2 no  1

VEL=  0.50        C  D  E  6          0.5201E+25
VEL=  1.00        C  D  E  6          0.4710E+25
VEL=  1.50        C  D  E  6          0.3928E+25
VEL=  2.00        C  D  E  6          0.2912E+25
VEL=  2.50        C  D  E  6          0.1738E+25
VEL=  3.00        C  D  E  6          0.5082E+24
VEL=  3.50        C  D        7      -0.6563E+24
VEL=  4.00        C  D        7      -0.1611E+25
VEL=  4.50           D        7      -0.2187E+25
VEL=  5.00           D        7      -0.2198E+25
VEL=  5.50           D        7      -0.1431E+25
VEL=  6.00                            0.3481E+24
VEL=  6.50                            0.3394E+25
VEL=  7.00                            0.7986E+25
VEL=  7.50                            0.1442E+26
VEL=  8.00                            0.2303E+26
VEL=  8.50                            0.3415E+26
VEL=  9.00                            0.4815E+26
VEL=  9.50                            0.6542E+26
VEL= 10.00                            0.8638E+26
VEL= 10.50

 200.0                    mr, lbm                 1
 20.0                     lr, in                  2
 50.0                     hr, in                  3
 100000.0                 Ryp, lbm in^2           4
 20000.0                  Rzp, lbm in^2           5
 60.0                     alphar, degrees         6
 41.0                     cw, in                  7
 0.0                      cf, in                  8
 0.0                      lambda, degrees         9
 10.0                     mf, lbm                 10
 2.0                      d, in                   11
 30.00                    u, in                   12
 50.0                     Fyp, lbm in^2           13
 2000.0                   Fzp, lbm in^2           14
 45.00                    alphaf, degrees         15
 50.0                     CrR, lbm in             16
 50.0                     CfR, lbm in             17
```

Computer output for figure 5.11. Model approximating a Schwinn Varsity bicycle and rider is self-stable between 8.0 and 12.0 mph.

```
A =    3.95152640E+08   > 0
B =    3.22249620E+07   V  > 0
C =  -9.77952973E+09 +  803181.00000000  V^2  > 0
D =  -1.62276576E+08 V +   12427.99218750  V^3  > 0
E =   4.79032238E+10 +  -1.09805950E+06  V^2  > 0
6th=   2.09715242E+13 V^3 +  -2.51020961E+17 V  > 0
7th=   2.60633923E+17 V^4 +  -5.38259588E+21 V^2 +  -9.01019454E+24 > 0
Want to know stable velocity regions? 1 yes 2 no  1

VEL=  0.50     C  D    6  7      -0.9425E+25
VEL=  1.00     C  D    6  7      -0.1065E+26
VEL=  1.50     C  D    6  7      -0.1264E+26
VEL=  2.00     C  D    6  7      -0.1528E+26
VEL=  2.50     C  D    6  7      -0.1845E+26
VEL=  3.00     C  D    6  7      -0.2199E+26
VEL=  3.50     C  D    6  7      -0.2568E+26
VEL=  4.00     C  D    6  7      -0.2929E+26
VEL=  4.50     C  D    6  7      -0.3252E+26
VEL=  5.00     C  D    6  7      -0.3506E+26
VEL=  5.50     C  D    6  7      -0.3656E+26
VEL=  6.00     C  D    6  7      -0.3662E+26
VEL=  6.50                7      -0.3481E+26
VEL=  7.00                7      -0.3066E+26
VEL=  7.50                7      -0.2367E+26
VEL=  8.00                7      -0.1328E+26
VEL=  8.50                        0.1071E+25
VEL=  9.00                        0.2002E+26
VEL=  9.50                        0.4421E+26
VEL= 10.00                        0.7434E+26
VEL= 10.50                        0.1111E+27
VEL= 11.00                        0.1554E+27
VEL= 11.50                        0.2079E+27
VEL= 12.00          E             0.2695E+27
VEL= 12.50          E             0.3410E+27
VEL= 13.00          E             0.4235E+27
VEL= 13.50
  200.0                      mr, lbm                 1
  20.0                       lr, in                  2
  50.0                       hr, in                  3
  100000.0                   Ryp, lbm in^2           4
  20000.0                    Rzp, lbm in^2           5
  60.0                       alphar, degrees         6
  41.0                       cw, in                  7
  1.00                       cf, in                  8
  15.0                       lambda, degrees         9
  10.0                       mf, lbm                 10
  2.0                        d, in                   11
  30.00                      u, in                   12
  50.0                       Fyp, lbm in^2           13
  2000.0                     Fzp, lbm in^2           14
  45.00                      alphaf, degrees         15
  50.0                       CrR, lbm in             16
  50.0                       CfR, lbm in             17
```

Computer output for figure 5.12. The model is self-stable after 3.5 mph. Figure 5.12 shows a positive d and positive c_f for this model. However, the correct figure would show negative d and c_f equal to zero.

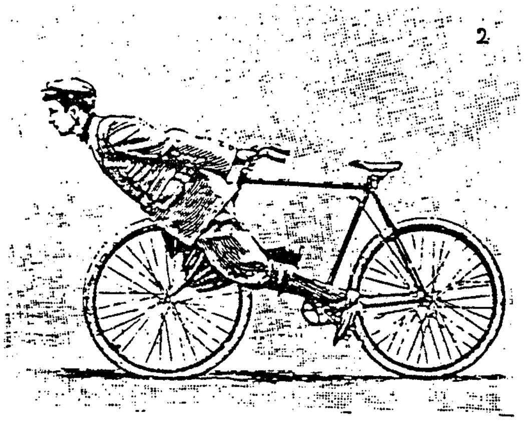
```
A =    5.56601080E+07   > 0
B =    1.97555212E+06   V  > 0
C =  -3.25146848E+08 +  119030.49218750  V^2  > 0
D =    1.95284475E+06 V +    0.00000000E-01  V^3  > 0
E =    1.08488152E+08 +  -3.72743979E-02  V^2  > 0
6th=   2.35150934E+11 V^3 +  -7.51040127E+14 V  > 0
7th=   0.00000000E-01 V^4 +   4.59213402E+17 V^2 +  -1.89007294E+21 > 0
Want to know stable velocity regions? 1 yes 2 no  1

VEL=  0.50        C        6  7      -0.1855E+22
VEL=  1.00        C        6  7      -0.1748E+22
VEL=  1.50        C        6  7      -0.1570E+22
VEL=  2.00        C        6  7      -0.1321E+22
VEL=  2.50        C        6  7      -0.1001E+22
VEL=  3.00                 6  7      -0.6099E+21
VEL=  3.50                    7      -0.1476E+21
VEL=  4.00                            0.3859E+21
VEL=  4.50                            0.9904E+21
VEL=  5.00                            0.1666E+22
VEL=  5.50                            0.2413E+22
VEL=  6.00                            0.3231E+22
VEL=  6.50                            0.4120E+22
VEL=  7.00                            0.5080E+22
VEL=  7.50                            0.6111E+22
VEL=  8.00                            0.7214E+22
VEL=  8.50                            0.8387E+22
VEL=  9.00                            0.9632E+22
VEL=  9.50                            0.1095E+23
VEL= 10.00                            0.1233E+23
VEL= 10.50                            0.1379E+23
VEL= 11.00
```

```
 200.0                     mr, lbm                 1
 20.0                      lr, in                  2
 70.0                      hr, in                  3
 00000.0                   Ryp, lbm in^2           4
 12500.0                   Rzp, lbm in^2           5
 45.0                      alphar, degrees         6
 40.0                      cw, in                  7
  0.0                      cf, in                  8
 -1.0                      lambda, degrees         9
  5.0                      mf, lbm                 10
 -0.6                      d, in                   11
 10.00                     u, in                   12
 0000.0                    Fyp, lbm in^2           13
 5000.0                    Fzp, lbm in^2           14
 5.00                      alphaf, degrees         15
 00.0                      CrR, lbm in             16
 00.0                      CfR, lbm in             17
```

Computer output for figure 5.13. The model is self-stable after 7.5 mph.

```
A =    6.20933184E+08   > 0
B =    1.69176500E+07   V   > 0
C =   -4.73024614E+09 +   613096.00000000   V^2  > 0
D =   -6.04941920E+07 V +    12048.56152344   V^3   > 0
E =    3.56813338E+09 +    86333.79687500   V^2   > 0
6th=    2.89079191E+12 V^3 +  -4.24617947E+16 V   > 0
7th=    3.48298848E+16 V^4 +  -7.11189031E+20 V^2 +    1.54746767E+24 > 0
Want to know stable velocity regions? 1 yes 2 no  1

VEL=  0.50        C  D     6        0.1493E+25
VEL=  1.00        C  D     6        0.1331E+25
VEL=  1.50        C  D     6        0.1069E+25
VEL=  2.00        C  D     6        0.7197E+24
VEL=  2.50        C  D     6        0.3012E+24
VEL=  3.00        C  D     6   7   -0.1645E+24
VEL=  3.50        C  D     6   7   -0.6497E+24
VEL=  4.00        C  D     6   7   -0.1122E+25
VEL=  4.50        C        6   7   -0.1543E+25
VEL=  5.00                 6   7   -0.1871E+25
VEL=  5.50                 6   7   -0.2058E+25
VEL=  6.00                 6   7   -0.2052E+25
VEL=  6.50                 6   7   -0.1794E+25
VEL=  7.00                     7   -0.1223E+25
VEL=  7.50                     7   -0.2701E+24
VEL=  8.00                          0.1137E+25
VEL=  8.50                          0.3076E+25
VEL=  9.00                          0.5630E+25
VEL=  9.50                          0.8886E+25
VEL= 10.00                          0.1294E+26
VEL= 10.50
```

```
200.0          mr, lbm               1
20.0           lr, in                2
50.0           hr, in                3
100000.0       Ryp, lbm in^2         4
20000.0        Rzp, lbm in^2         5
60.0           alphar, degrees       6
41.0           cw, in                7
-1.0           cf, in                8
-2.0           lambda, degrees       9
10.0           mf, lbm               10
2.0            d, in                 11
30.00          u, in                 12
50.0           Fyp, lbm in^2         13
2000.0         Fzp, lbm in^2         14
45.00          alphaf, degrees       15
50.0           CrR, lbm in           16
50.0           CfR, lbm in           17
```



APPENDIX D

BICYCLE STABILITY DEMONSTRATION ANNOUNCEMENT

BICYCLE STABILITY DEMONSTRATION

On Saturday november 15th at 11:00am Scott Hand will demonstrate some results of his bicycle stability research in the west end of the veternarian parking lot. This demonstration is restricted to the weekends due to the availibility of the parking. The demo will show:

1) A riderless bicycle can be stable. We will push a schwinn varsity, and show that it stays upright and goes straight for speeds greater that 7mph. Even after perturbing its motion the bicycle will return to an upright position.(or constant curvature.
2) Wheel alignment is critical to the straightness or curvature of stable motion. We will misalign a wheel and show that the bicycle takes a curved path. The direction of this curved path is predictable.
3) Increasing the inertia of the front wheel of a bicycle significantly increases its stability. We will substitute a much heavier wheel and show slower stable speeds and increased ability to recover from a pertubation.
4) A bicycle with negative trail can be stabilized by adding a negative spring. We will extend the fork to get a negative trail and show this configuration is unstable. We will then add a negative spring to the steering axis and show how this configuration is stable.
5) A riderless bicycle can naturally self right itself once in a steady turn. This is a nonlinear phenomena in which a bicycle stable in straight line motion can also be stable in a steady curve, but as it slows will naturally upright into a straight line position.
6) A bicycle can be towed by a string (time permitting). We will show how to tow a bicycle, similar to flying a kite. In doing so we will show that the placement of attachment ot the string is critical to have stability.

Those attending are welcome to try any of the experiments after the demonstration. For further information please call Scott Hand at 255-3518 or Jim Papadopoulos at 255-5035.

Rain date Sunday November 16th at 1:00pm.

Please see map to the parking lot on reverse side.

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