Derivation of the Linearized Equations for an Uncontrolled Bicycle

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Abstract: The linearized equations of motion for an uncontrolled bicycle are derived. The bicycle has its usual construction and the contact between the wheels and the road is modelled by holonomic constraints in the normal direction and non-holonomic constraints of no-slip in the tangential directions. The bicycle rides on a plane road surface that is inclined without superelevation. All friction is neglected, apart from the friction needed to enforce the non-holonomic constraints, but the bicycle may be driven or braked by moments at both wheels.

1. Introduction

In this report, the linearized equations for a bicycle are derived analytically. This is mainly done as an exercise in dynamics and as a check for previous calculations. The derivation is performed in a systematic way, so it is possible to extend the analysis and to include higher-order terms.

The first paper that gives the apparently correct equations is by Whipple [1].

The notation and the meaning of the symbols is in many ways the same as that in the conference paper [2].

2. Description of the bicycle model

A bicycle of the familiar construction is considered. The model for the bicycle consists of four rigid bodies that are interconnected by revolute joints. The four bodies are the rear frame with the rider rigidly attached to it, the rear wheel, the front frame including the front fork and handle bar, and the front wheel. The rear frame and the front frame are joined by the steering head; the rear wheel is connected to the rear frame by the revolute joint at the axle; and the front wheel is connected to the front fork by another revolute joint at its axle. The bicycle rides on a flat road with the rims of the wheels touching the road. The shape of the wheels is approximated by a thin rotationally symmetric disk with a knife edge at its rim. There is no slip between the road and the wheels.

In the nominal configuration, the origin O of the global orthogonal right-handed coordinate system $OXYZ$ is at the contact point between the rear wheel and the road. The Z-axis points in the downward direction perpendicular to the road surface, the X-axis points in the forward direction, and the Y-axis points to the right. The planes of symmetry of the frames and the planes of the wheels are all in the XZ-plane. The rear wheel has a radius R_{rw}, a mass m_{rw} and moments of inertia A_{xx}, A_{yy} and $A_{zz} = A_{xx}$; its centre of mass is at the centre of the wheel, which is at the position $\mathbf{x}_{\mathrm{rw}} = (0, 0, -R_{\mathrm{rw}})^T$. The centre of mass of the rear frame is at the position $\mathbf{x}_{\mathrm{rf}} = (x_{\mathrm{rf}}, 0, z_{\mathrm{rf}})^T$, its mass is m_{rf} and its inertia tensor with respect to its centre of mass is given by

$$\mathbf{B} = \begin{pmatrix} B_{xx} & 0 & B_{xz} \\ 0 & B_{yy} & 0 \\ B_{xz} & 0 & B_{zz} \end{pmatrix}. \tag{1}$$

Note that B_{xz} is minus the corresponding product of inertia. In the same way, the centre of mass of the front frame is located at $\mathbf{x}_{\mathrm{ff}} = (x_{\mathrm{ff}}, 0, z_{\mathrm{ff}})^T$, its mass is m_{ff} and its inertia tensor with respect to its centre of mass is given by

$$\mathbf{C} = \begin{pmatrix} C_{xx} & 0 & C_{xz} \\ 0 & C_{yy} & 0 \\ C_{xz} & 0 & C_{zz} \end{pmatrix}. \tag{2}$$

The wheelbase is w and the radius of the front wheel is R_{fw}, so the centre of the front wheel, which is also its centre of mass, is at $\mathbf{x}_{\mathrm{fw}} = (w, 0, -R_{\mathrm{fw}})^T$. Its mass is m_{fw} and its moments of inertia are D_{xx}, D_{yy} and $D_{zz} = D_{xx}$. The angle that the steering axis makes with the Z-axis is λ, while the trail of the front wheel is t. Also the contact points of the wheels with the road are introduced, $\mathbf{c}_{\mathrm{rw}} = (0, 0, 0)^T$ for the rear wheel and $\mathbf{c}_{\mathrm{fw}} = (w, 0, 0)^T$ for the front wheel. Furthermore, a point on the steering axis on the road surface, which will be called the steering point, $\mathbf{x}_{\mathrm{sp}} = (w + t, 0, 0)^T$, and a unit vector along the steering axis, $\boldsymbol{\lambda} = (\sin\lambda, 0, \cos\lambda)^T$, are introduced. It is assumed that the bicycle rides on a slope, so the acceleration of gravity has the components $\mathbf{g} = (g_X, 0, g_Z)^T$. Braking moments, M_{rw} and M_{fw}, are applied to the rear and front wheels respectively.

In the previous paragraph, the coordinates were all given for the reference position. As the bicycle moves, these coordinates will change. Seven generalized coordinates are needed to describe the configuration of the system, which are chosen as follows. The position of the contact point of the rear wheel is described by two Cartesian coordinates in the plane of the road surface, x and y, with respect to the global coordinate system. The orientation of the rear frame is described by three angles, the yaw angle ψ about the Z-axis, the roll angle ϕ about the yawed longitudinal axis, and the pitch angle χ about the current lateral axis. The pitch angle is not independent, but is a function of the roll angle ϕ and the steering angle β (to be discussed below). Owing to the symmetry of the system, the linear terms in this dependence are zero. The relative rotation angle of the rear wheel with respect to the rear frame is denoted by θ_{rw}, the relative rotation angle of the front frame with respect to the rear frame is β, and the relative rotation angle of the front wheel with respect to the front frame is θ_{fw}. The system has three degrees of freedom; the independent generalized velocities are the rotation rate of the rear wheel, $\dot{\theta}_{\mathrm{rw}}$, the roll velocity, $\dot{\phi}$, and the steering velocity, $\dot{\beta}$.

It is convenient to introduce, besides the global reference frame, two local systems of coordinates: the body-fixed systems for the rear frame and the front frame respectively. Their origins are located in the centres of mass and, in the reference configuration, their axes are parallel to the axes of the global system. Vectors or coordinate vectors with respect to the system of the rear frame are indicated by an overbar, while those with respect to the system of the front frame are indicated by a hat.

3. Configuration analysis

Four of the seven generalized coordinates are cyclic; these are the two Cartesian coordinates of the contact point of the rear wheel, x and y, and the two rotation angles of the wheels, θ_{rw} and θ_{fw}. All of these can be chosen to be zero at the currect time without loss of generality. The yaw angle of the rear frame, ψ, is only cyclic if the longitudinal component of the acceleration of gravity is zero, that is, if the bicycle rides on a level surface, but even then this angle appears in the kinematic differential equations.

We restrict ourselves to the trivial motion, in which the bicycle is running straight ahead, and small deviations from this nominal motion. First the trivial motion is analysed. The out-of-plane variables, y, ψ, ϕ and β and their time derivatives are all zero, as is the dependent pitch angle χ. The rotations of the wheels are coupled to the travelled distance x by the relations $\theta_{\mathrm{rw}} = -R_{\mathrm{rw}}x$ and $\theta_{\mathrm{fw}} = -R_{\mathrm{fw}}x$. With this, the equation of motion becomes

$$(m_{\mathrm{t}}R_{\mathrm{rw}}^2 + A_{yy} + D_{yy}R_{\mathrm{rw}}^2/R_{\mathrm{fw}}^2)\ddot{\theta}_{\mathrm{rw}} = M_{\mathrm{rw}} + M_{\mathrm{fw}}R_{\mathrm{rw}}/R_{\mathrm{fw}} - m_{\mathrm{t}}R_{\mathrm{rw}}g_X, \tag{3}$$

where the total mass m_{t},

$$m_{\mathrm{t}} = m_{\mathrm{rw}} + m_{\mathrm{rf}} + m_{\mathrm{ff}} + m_{\mathrm{fw}} \tag{4}$$

has been introduced. This leads to a solution for the forward speed v that is a function of the time τ, so $\dot{x} = v(\tau)$, $\dot{\theta}_{\mathrm{rw}} = -v(\tau)/R_{\mathrm{rw}}$ and $\dot{\theta}_{\mathrm{fw}} = -v(\tau)/R_{\mathrm{fw}}$.

For the linearized equations for small perturbations from a trivial solution, because of the symmetry of the bicycle, the in-plane motion and the out-of-plane motion are decoupled. As the perturbation of the in-plane motion leads only to a perturbed trivial motion, our interest will be focused on the out-of-plane motion.

The orientation and the position of the characteristic points on the four bodies of the bicycle can now be obtained. The orientation is the absolute orientation and is expressed as a rotation vector for the small rotations of the rear and front frames. Small rotation vectors are the same in the linear approximation in any of the three coordinate systems. For the rear frame, the rotation vector is

$$\boldsymbol{\phi}_{\mathrm{rw}} = \bar{\boldsymbol{\phi}}_{\mathrm{rw}} = \begin{pmatrix} \phi \\ 0 \\ \psi \end{pmatrix}. \tag{5}$$

The contact point of the rear wheel with the road obviously has the coordinates

$$\mathbf{c}_{\mathrm{rw}} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

The centre of the rear wheel has the coordinates

$$\begin{aligned} \mathbf{x}_{\mathrm{rw}} &= \mathbf{c}_{\mathrm{rw}} + (\mathbf{I} + \tilde{\boldsymbol{\phi}}_{\mathrm{rw}})(\bar{\mathbf{x}}_{\mathrm{rw}} - \bar{\mathbf{c}}_{\mathrm{rw}}) \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -\psi & 0 \\ \psi & 1 & -\phi \\ 0 & \phi & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -R_{\mathrm{rw}} \end{pmatrix} = \begin{pmatrix} 0 \\ R_{\mathrm{rw}}\phi \\ -R_{\mathrm{rw}} \end{pmatrix} \end{aligned} \tag{7}$$

The tilde denotes the antimetric matrix associated with the rotation vector. The position of the centre of mass of the rear frame is given by

$$\begin{aligned} \mathbf{x}_{\mathrm{rf}} &= \mathbf{c}_{\mathrm{rw}} + (\mathbf{I} + \tilde{\boldsymbol{\phi}}_{\mathrm{rw}})(\bar{\mathbf{x}}_{\mathrm{rf}} - \bar{\mathbf{c}}_{\mathrm{rw}}) \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -\psi & 0 \\ \psi & 1 & -\phi \\ 0 & \phi & 1 \end{pmatrix} \begin{pmatrix} x_{\mathrm{rf}} \\ 0 \\ z_{\mathrm{rf}} \end{pmatrix} = \begin{pmatrix} x_{\mathrm{rf}} \\ -z_{\mathrm{rf}}\phi + x_{\mathrm{rf}}\psi \\ z_{\mathrm{rf}} \end{pmatrix}. \end{aligned} \tag{8}$$

Now we proceed with the front frame. The position of the steering point is given by

$$\begin{aligned} \mathbf{x}_{\mathrm{sp}} &= \mathbf{c}_{\mathrm{rw}} + (\mathbf{I} + \tilde{\boldsymbol{\phi}}_{\mathrm{rw}})(\bar{\mathbf{x}}_{\mathrm{sp}} - \bar{\mathbf{c}}_{\mathrm{rw}}) \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -\psi & 0 \\ \psi & 1 & -\phi \\ 0 & \phi & 1 \end{pmatrix} \begin{pmatrix} w+t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w+t \\ (w+t)\psi \\ 0 \end{pmatrix} \end{aligned} \tag{9}$$

The front frame rotates over an angle β with respect to the rear frame about the inclined steering axis, which makes an angle of λ with respect to the local $\bar{z}$-axis of the rear frame. So the rotation vector of the front frame is

$$\boldsymbol{\phi}_{\mathrm{ff}} = \bar{\boldsymbol{\phi}}_{\mathrm{ff}} = \hat{\boldsymbol{\phi}}_{\mathrm{ff}} = \begin{pmatrix} \phi + \beta\sin\lambda \\ 0 \\ \psi + \beta\cos\lambda \end{pmatrix}. \tag{10}$$

In the linear approximation, the contact point between the front wheel and the road has fixed coordinates in the frame of the front wheel. This follows from the symmetry, but it can also be verified directly. So this contact point is at the position

$$\begin{aligned} \mathbf{c}_{\mathrm{fw}} &= \mathbf{x}_{\mathrm{sp}} + (\mathbf{I} + \tilde{\boldsymbol{\phi}}_{\mathrm{ff}})(\hat{\mathbf{c}}_{\mathrm{fw}} - \hat{\mathbf{x}}_{\mathrm{sp}}) \\ &= \begin{pmatrix} w+t \\ (w+t)\psi \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -\psi - \beta\cos\lambda & 0 \\ \psi + \beta\cos\lambda & 1 & -\phi - \beta\sin\lambda \\ 0 & \phi + \beta\sin\lambda & 1 \end{pmatrix} \begin{pmatrix} -t \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} w \\ w\psi - t\beta\cos\lambda \\ 0 \end{pmatrix}. \end{aligned} \tag{11}$$

From this we see that the Z-coordinate of the contact point is zero to first order, so the constraint in normal direction is satisfied and our statement that the pitch angle is zero is confirmed. The centre of the front wheel is at the position

$$\begin{aligned}
\mathbf{x}_{\mathrm{fw}} &= \mathbf{x}_{\mathrm{sp}} + (\mathbf{I} + \tilde{\boldsymbol{\phi}}_{\mathrm{ff}})(\hat{\mathbf{x}}_{\mathrm{fw}} - \hat{\mathbf{x}}_{\mathrm{sp}}) \\
&= \begin{pmatrix} w+t \\ (w+t)\psi \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -\psi-\beta\cos\lambda & 0 \\ \psi+\beta\cos\lambda & 1 & -\phi-\beta\sin\lambda \\ 0 & \phi+\beta\sin\lambda & 1 \end{pmatrix} \begin{pmatrix} -t \\ 0 \\ -R_{\mathrm{fw}} \end{pmatrix} \\
&= \begin{pmatrix} w \\ R_{\mathrm{fw}}\phi + w\psi + u_{\mathrm{fw}}\beta \\ -R_{\mathrm{fw}} \end{pmatrix}.
\end{aligned} \tag{12}$$

Here, we have introduced

$$u_{\mathrm{fw}} = R_{\mathrm{fw}} \sin\lambda - t\cos\lambda, \tag{13}$$

the distance of the centre of the front wheel to the steering axis, positive if this centre is in front of the steering axis. The centre of mass of the front frame is at the position

$$\begin{aligned}
\mathbf{x}_{\mathrm{ff}} &= \mathbf{x}_{\mathrm{sp}} + (\mathbf{I} + \tilde{\boldsymbol{\phi}}_{\mathrm{ff}})(\hat{\mathbf{x}}_{\mathrm{ff}} - \hat{\mathbf{x}}_{\mathrm{sp}}) \\
&= \begin{pmatrix} w+t \\ (w+t)\psi \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & -\psi-\beta\cos\lambda & 0 \\ \psi+\beta\cos\lambda & 1 & -\phi-\beta\sin\lambda \\ 0 & \phi+\beta\sin\lambda & 1 \end{pmatrix} \begin{pmatrix} x_{\mathrm{ff}}-w-t \\ 0 \\ z_{\mathrm{ff}} \end{pmatrix} \\
&= \begin{pmatrix} x_{\mathrm{ff}} \\ -z_{\mathrm{ff}}\phi + x_{\mathrm{ff}}\psi + u_{\mathrm{ff}}\beta \\ z_{\mathrm{ff}} \end{pmatrix}.
\end{aligned} \tag{14}$$

Here, we have introduced

$$u_{\mathrm{ff}} = -z_{\mathrm{ff}} \sin\lambda + (x_{\mathrm{ff}} - w - t)\cos\lambda, \tag{15}$$

the distance of the centre of mass of the front fork to the steering axis, positive if this point is in front of the steering axis.

4. Virtual displacements

The analysis of the virtual displacements is a little more involved than the analysis of the displacements, velocities and accelerations, because constant as well as linear terms have to be retained in the coefficients of the virtual variations of the degrees of freedom, so these cannot simply be obtained from differentiating the linearized expressions for the displacements or velocities. The inclusion of the small terms yields the so-called geometric stiffness in the equations of motion. We start from a configuration that is a little displaced from the nominal configuration as described in the previous section. At first, all coordinates are considered to be independent; the constraints are imposed in the course of the calculations.

The rear frame and the rear wheel have the virtual rotations, expressed in the system of the rear frame, of

$$\delta\bar{\boldsymbol{\phi}}_{\mathrm{rf}} = \begin{pmatrix} \delta\phi \\ \phi\delta\psi + \delta\chi \\ \delta\psi \end{pmatrix}, \qquad \delta\bar{\boldsymbol{\phi}}_{\mathrm{rw}} = \begin{pmatrix} \delta\phi \\ \phi\delta\psi + \delta\chi + \delta\theta_{\mathrm{rw}} \\ \delta\psi \end{pmatrix}. \tag{16}$$

It should be noted that $\delta\chi$ is dependent and small. It is further convenient to introduce a virtual rotation that does not involve the variation of the pitch angle as

$$\delta\bar{\boldsymbol{\phi}}'_{\mathrm{rf}} = \begin{pmatrix} \delta\phi \\ \phi\delta\psi \\ \delta\psi \end{pmatrix}. \tag{17}$$

The virtual displacement of the contact point of the rear wheel with the road, expressed in the system of the rear frame, is

$$\delta\bar{\mathbf{c}}_{\mathrm{rw}} = (\mathbf{I} - \tilde{\boldsymbol{\phi}}_{\mathrm{rf}})\delta\mathbf{c}_{\mathrm{rw}} = \begin{pmatrix} 1 & \psi & 0 \\ -\psi & 1 & \phi \\ 0 & -\phi & 1 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ 0 \end{pmatrix} = \begin{pmatrix} \delta x + \psi\delta y \\ -\psi\delta x + \delta y \\ -\phi\delta y \end{pmatrix}. \tag{18}$$

The virtual displacement of the centre of the rear wheel is

$$\begin{gathered} \delta\bar{\mathbf{x}}_{\mathrm{rw}} = \delta\bar{\mathbf{c}}_{\mathrm{rw}} + \delta\bar{\boldsymbol{\phi}}'_{\mathrm{rf}} \times (\bar{\mathbf{x}}_{\mathrm{rw}} - \bar{\mathbf{c}}_{\mathrm{rw}}) \\ = \begin{pmatrix} \delta x + \psi\delta y \\ -\psi\delta x + \delta y \\ -\phi\delta y \end{pmatrix} + \begin{pmatrix} \delta\phi \\ \phi\delta\psi \\ \delta\psi \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -R_{\mathrm{rw}} \end{pmatrix} = \begin{pmatrix} \delta x + \psi\delta y - R_{\mathrm{rw}}\phi\delta\psi \\ -\psi\delta x + \delta y + R_{\mathrm{rw}}\delta\phi \\ -\phi\delta y \end{pmatrix}. \end{gathered} \tag{19}$$

The virtual dispalcement of the material point that is in contact with the ground has to be zero,

$$\begin{gathered} \mathbf{0} = \delta\bar{\mathbf{x}}_{\mathrm{rw}} + \delta\bar{\boldsymbol{\phi}}_{\mathrm{rw}} \times (\bar{\mathbf{c}}_{\mathrm{rw}} - \bar{\mathbf{x}}_{\mathrm{rw}}) \\ = \begin{pmatrix} \delta x + \psi\delta y - R_{\mathrm{rw}}\phi\delta\psi \\ -\psi\delta x + \delta y + R_{\mathrm{rw}}\delta\phi \\ -\phi\delta y \end{pmatrix} + \begin{pmatrix} \delta\phi \\ \phi\delta\psi + \delta\chi + \delta\theta_{\mathrm{rw}} \\ \delta\psi \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ R_{\mathrm{rw}} \end{pmatrix} \\ = \begin{pmatrix} \delta x + \psi\delta y + R_{\mathrm{rw}}\delta\chi + R_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} \\ -\psi\delta x + \delta y \\ -\phi\delta y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{gathered} \tag{20}$$

From this, the relations for the virtual displacements are found as

$$\delta x = -R_{\mathrm{rw}}(\delta\chi + \delta\theta_{\mathrm{rw}}), \qquad \delta y = \psi\delta x = -R_{\mathrm{rw}}\psi\delta\theta_{\mathrm{rw}}, \tag{21}$$

from which follows that δy is small. The virtual displacement of the centre of the rear wheel can be simplified to

$$\delta\bar{\mathbf{x}}_{\mathrm{rw}} = \begin{pmatrix} -R_{\mathrm{rw}}(\phi\delta\psi + \delta\chi + \delta\theta_{\mathrm{rw}}) \\ R_{\mathrm{rw}}\delta\phi \\ 0 \end{pmatrix}. \tag{22}$$

At a later stage, the variation of the yaw angle and pitch angle will be removed. The virtual displacement of the steering point becomes

$$\begin{aligned} \delta\bar{\mathbf{x}}_{\mathrm{sp}} &= \delta\bar{\mathbf{x}}_{\mathrm{rw}} + \delta\bar{\boldsymbol{\phi}}_{\mathrm{rf}} \times (\bar{\mathbf{x}}_{\mathrm{sp}} - \bar{\mathbf{x}}_{\mathrm{rw}}) \\ &= \begin{pmatrix} -R_{\mathrm{rw}}(\phi\delta\psi + \delta\chi + \delta\theta_{\mathrm{rw}}) \\ R_{\mathrm{rw}}\delta\phi \\ 0 \end{pmatrix} + \begin{pmatrix} \delta\phi \\ \phi\delta\psi + \delta\chi \\ \delta\psi \end{pmatrix} \times \begin{pmatrix} w+t \\ 0 \\ R_{\mathrm{rw}} \end{pmatrix} \\ &= \begin{pmatrix} -R_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} \\ (w+t)\delta\psi \\ -(w+t)(\phi\delta\psi + \delta\chi) \end{pmatrix}. \end{aligned} \tag{23}$$

If this is transformed to the system of the front frame, we obtain

$$\begin{aligned} \delta\hat{\mathbf{x}}_{\mathrm{sp}} = (\mathbf{I} - \beta\tilde{\boldsymbol{\lambda}})\delta\bar{\mathbf{x}}_{\mathrm{sp}} &= \begin{pmatrix} 1 & \beta\cos\lambda & 0 \\ -\beta\cos\lambda & 1 & \beta\sin\lambda \\ 0 & -\beta\sin\lambda & 1 \end{pmatrix} \begin{pmatrix} -R_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} \\ (w+t)\delta\psi \\ -(w+t)(\phi\delta\psi + \delta\chi) \end{pmatrix} \\ &= \begin{pmatrix} (w+t)\beta\delta\psi\cos\lambda - R_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} \\ (w+t)\delta\psi + R_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}}\cos\lambda \\ -(w+t)[(\phi + \beta\sin\lambda)\delta\psi + \delta\chi] \end{pmatrix}. \end{aligned} \tag{24}$$

The virtual rotation of the front frame is found to be

$$\begin{aligned} \delta\hat{\boldsymbol{\phi}}_{\mathrm{ff}} &= (\mathbf{I} - \beta\tilde{\boldsymbol{\lambda}})\delta\bar{\boldsymbol{\phi}}_{\mathrm{rf}} + \boldsymbol{\lambda}\delta\beta \\ &= \begin{pmatrix} 1 & \beta\cos\lambda & 0 \\ -\beta\cos\lambda & 1 & \beta\sin\lambda \\ 0 & -\beta\sin\lambda & 1 \end{pmatrix} \begin{pmatrix} \delta\phi \\ \phi\delta\psi + \delta\chi \\ \delta\psi \end{pmatrix} + \begin{pmatrix} \delta\beta\sin\lambda \\ 0 \\ \delta\beta\cos\lambda \end{pmatrix} \\ &= \begin{pmatrix} \delta\phi + \delta\beta\sin\lambda \\ (\phi + \beta\sin\lambda)\delta\psi - \beta\delta\phi\cos\lambda + \delta\chi \\ \delta\psi + \delta\beta\cos\lambda \end{pmatrix}, \end{aligned} \tag{25}$$

while the virtual rotation of the front wheel is

$$\delta\hat{\boldsymbol{\phi}}_{\mathrm{fw}} = \begin{pmatrix} \delta\phi + \delta\beta\sin\lambda \\ (\phi + \beta\sin\lambda)\delta\psi - \beta\delta\phi\cos\lambda + \delta\chi + \delta\theta_{\mathrm{fw}} \\ \delta\psi + \delta\beta\cos\lambda \end{pmatrix}. \tag{26}$$

The virtual displacement of the centre of the front wheel is

$$
\begin{aligned}
\delta\hat{\mathbf{x}}_{\text{fw}} &= \delta\hat{\mathbf{x}}_{\text{sp}} + \delta\hat{\boldsymbol{\phi}}_{\text{ff}} \times (\hat{\mathbf{x}}_{\text{fw}} - \hat{\mathbf{x}}_{\text{sp}}) \\
&= \begin{pmatrix} (w+t)\beta\delta\psi\cos\lambda - R_{\text{rw}}\delta\theta_{\text{rw}} \\ (w+t)\delta\psi + R_{\text{rw}}\beta\delta\theta_{\text{rw}}\cos\lambda \\ -(w+t)[(\phi+\beta\sin\lambda)\delta\psi + \delta\chi] \end{pmatrix} \\
&+ \begin{pmatrix} \delta\phi + \delta\beta\sin\lambda \\ (\phi+\beta\sin\lambda)\delta\psi - \beta\delta\phi\cos\lambda + \delta\chi \\ \delta\psi + \delta\beta\cos\lambda \end{pmatrix} \times \begin{pmatrix} -t \\ 0 \\ -R_{\text{fw}} \end{pmatrix} \\
&= \begin{pmatrix} [(w+t)\beta\cos\lambda - R_{\text{fw}}(\phi+\beta\sin\lambda)]\delta\psi + R_{\text{fw}}\beta\delta\phi\cos\lambda - R_{\text{fw}}\delta\chi - R_{\text{rw}}\delta\theta_{\text{rw}} \\ w\delta\psi + R_{\text{fw}}\delta\phi + u_{\text{fw}}\delta\beta + R_{\text{rw}}\beta\delta\theta_{\text{rw}}\cos\lambda \\ -w(\phi+\beta\sin\lambda)\delta\psi - t\beta\delta\phi\cos\lambda - w\delta\chi \end{pmatrix}.
\end{aligned} \tag{27}
$$

The virtual displacement of the contact point of the front wheel with the ground has to be zero,

$$
\begin{aligned}
\mathbf{0} &= \delta\hat{\mathbf{x}}_{\text{fw}} + \delta\hat{\boldsymbol{\phi}}_{\text{fw}} \times (\hat{\mathbf{c}}_{\text{fw}} - \hat{\mathbf{x}}_{\text{fw}}) \\
&= \begin{pmatrix} [(w+t)\beta\cos\lambda - R_{\text{fw}}(\phi+\beta\sin\lambda)]\delta\psi + R_{\text{fw}}\beta\delta\phi\cos\lambda - R_{\text{fw}}\delta\chi - R_{\text{rw}}\delta\theta_{\text{rw}} \\ w\delta\psi + R_{\text{fw}}\delta\phi + u_{\text{fw}}\delta\beta + R_{\text{rw}}\beta\delta\theta_{\text{rw}}\cos\lambda \\ -w(\phi+\beta\sin\lambda)\delta\psi - t\beta\delta\phi\cos\lambda - w\delta\chi \end{pmatrix} \\
&+ \begin{pmatrix} \delta\phi + \delta\beta\sin\lambda \\ (\phi+\beta\sin\lambda)\delta\psi - \beta\delta\phi\cos\lambda + \delta\chi + \delta\theta_{\text{fw}} \\ \delta\psi + \delta\beta\cos\lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ R_{\text{fw}} \end{pmatrix} \\
&= \begin{pmatrix} (w+t)\beta\delta\psi\cos\lambda - R_{\text{rw}}\delta\theta_{\text{rw}} + R_{\text{fw}}\delta\theta_{\text{fw}} \\ w\delta\psi - t\delta\beta\cos\lambda + R_{\text{rw}}\beta\delta\theta_{\text{rw}}\cos\lambda \\ -w(\phi+\beta\sin\lambda)\delta\psi - t\beta\delta\phi\cos\lambda - w\delta\chi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{28}
$$

From this we deduce that

$$
\begin{aligned}
\delta\psi &= f\delta\beta - R_{\text{rw}}\beta\delta\theta_{\text{rw}}\cos\lambda/w \\
\delta\chi &= -f\beta\delta\phi - f(\phi+\beta\sin\lambda)\delta\beta \\
\delta\theta_{\text{fw}} &= -f(w+t)\beta\delta\beta\cos\lambda/R_{\text{fw}} + \delta\theta_{\text{rw}}R_{\text{rw}}/R_{\text{fw}},
\end{aligned} \tag{29}
$$

where we have introduced the factor

$$
f = t\cos\lambda/w. \tag{30}
$$

With these expressions, the virtual displacements and rotations of the four bodies are found to be as follows. The virtual rotation of the rear wheel is

$$
\delta\bar{\boldsymbol{\phi}}_{\text{rw}} = \begin{pmatrix} \delta\phi \\ -f\beta\delta\phi - f\beta\delta\beta\sin\lambda + \delta\theta_{\text{rw}} \\ f\delta\beta - R_{\text{rw}}\beta\delta\theta_{\text{rw}}\cos\lambda/w \end{pmatrix}. \tag{31}
$$

The virtual dispalcement of the centre of the rear wheel is

$$\delta\bar{\mathbf{x}}_{\mathrm{rw}} = \begin{pmatrix} R_{\mathrm{rw}}(f\beta\delta\phi + f\beta\delta\beta\sin\lambda - \delta\theta_{\mathrm{rw}}) \\ R_{\mathrm{rw}}\delta\phi \\ 0 \end{pmatrix}. \tag{32}$$

The virtual rotation of the rear frame is

$$\delta\bar{\boldsymbol{\phi}}_{\mathrm{rf}} = \begin{pmatrix} \delta\phi \\ -f\beta\delta\phi - f\beta\delta\beta\sin\lambda \\ f\delta\beta - R_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}}\cos\lambda/w \end{pmatrix}. \tag{33}$$

The virtual displacement of the centre of mass of the rear frame is

$$\begin{gathered} \delta\bar{\mathbf{x}}_{\mathrm{rf}} = \delta\bar{\mathbf{x}}_{\mathrm{rw}} + \delta\bar{\boldsymbol{\phi}}_{\mathrm{rf}} \times (\bar{\mathbf{x}}_{\mathrm{rf}} - \bar{\mathbf{x}}_{\mathrm{rw}}) \\ = \begin{pmatrix} R_{\mathrm{rw}}(f\beta\delta\phi + f\beta\delta\beta\sin\lambda - \delta\theta_{\mathrm{rw}}) \\ R_{\mathrm{rw}}\delta\phi \\ 0 \end{pmatrix} + \begin{pmatrix} \delta\phi \\ -f\beta\delta\phi - f\beta\delta\beta\sin\lambda \\ f\delta\beta - R_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}}\cos\lambda/w \end{pmatrix} \times \begin{pmatrix} x_{\mathrm{rf}} \\ 0 \\ R_{\mathrm{rw}} + z_{\mathrm{rf}} \end{pmatrix} \\ = \begin{pmatrix} -fz_{\mathrm{rf}}\beta\delta\phi - fz_{\mathrm{rf}}\beta\delta\beta\sin\lambda - R_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} \\ -z_{\mathrm{rf}}\delta\phi + fx_{\mathrm{rf}}\delta\beta - x_{\mathrm{rf}}R_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}}\cos\lambda/w \\ fx_{\mathrm{rf}}\beta\delta\phi + fx_{\mathrm{rf}}\beta\delta\beta\sin\lambda \end{pmatrix}. \end{gathered} \tag{34}$$

The virtual rotation of the front frame is

$$\delta\hat{\boldsymbol{\phi}}_{\mathrm{ff}} = \begin{pmatrix} \delta\phi + \delta\beta\sin\lambda \\ -(\cos\lambda + f)\beta\delta\phi \\ (\cos\lambda + f)\delta\beta - R_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}}\cos\lambda/w \end{pmatrix}. \tag{35}$$

The virtual displacement of the centre of mass of the front frame is

$$\begin{gathered} \delta\hat{\mathbf{x}}_{\mathrm{ff}} = \delta\hat{\mathbf{x}}_{\mathrm{sp}} + \delta\hat{\boldsymbol{\phi}}_{\mathrm{ff}} \times (\hat{\mathbf{x}}_{\mathrm{ff}} - \hat{\mathbf{x}}_{\mathrm{sp}}) \\ = \begin{pmatrix} f(w+t)\beta\delta\beta\cos\lambda - R_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} \\ f(w+t)\delta\beta - fR_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}} \\ f(w+t)\beta\delta\phi \end{pmatrix} \\ + \begin{pmatrix} \delta\phi + \delta\beta\sin\lambda \\ -(\cos\lambda + f)\beta\delta\phi \\ (\cos\lambda + f)\delta\beta - R_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}}\cos\lambda/w \end{pmatrix} \times \begin{pmatrix} x_{\mathrm{ff}} - w - t \\ 0 \\ z_{\mathrm{ff}} \end{pmatrix} \\ = \begin{pmatrix} -z_{\mathrm{ff}}(\cos\lambda + f)\beta\delta\phi + f(w+t)\beta\delta\beta\cos\lambda - R_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} \\ -z_{\mathrm{ff}}\delta\phi + (u_{\mathrm{ff}} + fx_{\mathrm{ff}})\delta\beta - (x_{\mathrm{ff}} - w)R_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}}\cos\lambda/w \\ [(x_{\mathrm{ff}} - w - t)\cos\lambda + fx_{\mathrm{ff}}]\beta\delta\phi \end{pmatrix}. \end{gathered} \tag{36}$$

The virtual rotation of the front wheel is

$$\delta\hat{\boldsymbol{\phi}}_{\mathrm{fw}} = \begin{pmatrix} \delta\phi + \delta\beta\sin\lambda \\ -(\cos\lambda + f)\beta\delta\phi - f(w+t)\beta\delta\beta\cos\lambda/R_{\mathrm{fw}} + \delta\theta_{\mathrm{rw}}R_{\mathrm{rw}}/R_{\mathrm{fw}} \\ (\cos\lambda + f)\delta\beta - R_{\mathrm{rw}}\beta\delta\theta_{\mathrm{rw}}\cos\lambda/w \end{pmatrix}. \tag{37}$$

The virtual displacement of the centre of the front wheel is

$$\delta\hat{\mathbf{x}}_{\mathrm{fw}} = \begin{pmatrix} R_{\mathrm{fw}}(\cos\lambda + f)\beta\delta\phi + f(w+t)\beta\delta\beta\cos\lambda - R_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} \\ R_{\mathrm{fw}}\delta\phi + (u_{\mathrm{fw}} + fw)\delta\beta \\ 0 \end{pmatrix}. \tag{38}$$

The expressions may seem rather complicated; one has to remember that the small terms have only to be multiplied by the large terms in the equations of motion for the individual parts. In fact, the small terms in the $\bar{x}$, $\bar{z}$, $\hat{x}$ and $\hat{z}$ components of the virtual rotations and the $\bar{y}$ or $\hat{y}$ components of the virtual displacements are not used.

5. Velocity analysis

In this section we discuss the velocity analysis together with the non-holonomic constraints at the wheels. The velocities and angular velocities are expressed with respect to the system of the rear frame for the rear wheel and the rear frame and with respect to system of the front frame for the front frame and the front wheel. Because the system is scleronomic, the relations for the velocities can immediately be derived from the relations for the virtual displacements, by changing the virtual variations into time derivatives and dropping higher-order terms. The kinematic differential equations follow from the equations (21) and (29) as

$$\begin{aligned} \dot{x} &= -R_{\mathrm{rw}}\dot{\theta}_{\mathrm{rw}} = v, \\ \dot{y} &= -R_{\mathrm{rw}}\psi\dot{\theta}_{\mathrm{rw}} = v\psi, \\ \dot{\psi} &= f\dot{\beta} - R_{\mathrm{rw}}\beta\dot{\theta}_{\mathrm{rw}}\cos\lambda/w = f\dot{\beta} + v\beta\cos\lambda/w, \\ \dot{\chi} &= 0, \\ \dot{\theta}_{\mathrm{fw}} &= \dot{\theta}_{\mathrm{rw}}R_{\mathrm{rw}}/R_{\mathrm{fw}} = -v/R_{\mathrm{fw}}. \end{aligned} \tag{39}$$

The velocities follow directly from the equations (31–38). The angular velocity of the rear wheel is

$$\bar{\boldsymbol{\omega}}_{\mathrm{rw}} = \begin{pmatrix} \dot{\phi} \\ \dot{\theta}_{\mathrm{rw}} \\ f\dot{\beta} + v\beta\cos\lambda/w \end{pmatrix}. \tag{40}$$

The velocity of the centre of the rear wheel is

$$\dot{\bar{\mathbf{x}}}_{\mathrm{rw}} = \begin{pmatrix} v \\ R_{\mathrm{rw}}\dot{\phi} \\ 0 \end{pmatrix}. \tag{41}$$

The angular velocity of the rear frame is

$$\bar{\boldsymbol{\omega}}_{\mathrm{rf}} = \begin{pmatrix} \dot{\phi} \\ 0 \\ f\dot{\beta} + v\beta\cos\lambda/w \end{pmatrix}. \tag{42}$$

The velocity of the centre of mass of the rear frame is

$$\dot{\mathbf{x}}_{\mathrm{rf}} = \begin{pmatrix} v \\ -z_{\mathrm{rf}}\dot{\phi} + f x_{\mathrm{rf}}\dot{\beta} + x_{\mathrm{rf}} v\beta\cos\lambda/w \\ 0 \end{pmatrix}. \tag{43}$$

The angular velocity of the front frame is

$$\hat{\boldsymbol{\omega}}_{\mathrm{ff}} = \begin{pmatrix} \dot{\phi} + \dot{\beta}\sin\lambda \\ 0 \\ (\cos\lambda + f)\dot{\beta} + v\beta\cos\lambda/w \end{pmatrix}. \tag{44}$$

The velocity of the centre of mass of the front frame is

$$\dot{\mathbf{x}}_{\mathrm{ff}} = \begin{pmatrix} v \\ -z_{\mathrm{ff}}\dot{\phi} + (u_{\mathrm{ff}} + f x_{\mathrm{ff}})\dot{\beta} + (x_{\mathrm{ff}} - w) v\beta\cos\lambda/w \\ 0 \end{pmatrix}. \tag{45}$$

The angular velocity of the front wheel is

$$\hat{\boldsymbol{\omega}}_{\mathrm{fw}} = \begin{pmatrix} \dot{\phi} + \dot{\beta}\sin\lambda \\ \dot{\theta}_{\mathrm{rw}} R_{\mathrm{rw}}/R_{\mathrm{fw}} \\ (\cos\lambda + f)\dot{\beta} + v\beta\cos\lambda/w \end{pmatrix}. \tag{46}$$

The velocity of the centre of the front wheel is

$$\dot{\mathbf{x}}_{\mathrm{fw}} = \begin{pmatrix} v \\ R_{\mathrm{fw}}\dot{\phi} + (u_{\mathrm{fw}} + f w)\dot{\beta} \\ 0 \end{pmatrix}. \tag{47}$$

6. Acceleration analysis

In this section we discuss the acceleration analysis. In the equations of motion, not the angular accelerations, but the time derivatives of the components of the angular velocity are needed. If the angular velocity is decomposed along body-fixed components, as is the case for the rear and front frame, these derivatives are equal to the components of the angular velocity. This is no longer true for the rear and front wheels, however.

By taking time derivatives of the kinematic differential equations (39), we obtain the relations

$$\begin{aligned} \ddot{x} &= \dot{v}, \\ \ddot{y} &= v\dot{\psi} + \dot{v}\psi = v f\dot{\beta} + v^2\beta\cos\lambda/w + \dot{v}\psi, \\ \ddot{\psi} &= f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \\ \ddot{\chi} &= 0, \\ \ddot{\theta}_{\mathrm{fw}} &= \ddot{\theta}_{\mathrm{rw}} R_{\mathrm{rw}}/R_{\mathrm{fw}} = -\dot{v}/R_{\mathrm{fw}}. \end{aligned} \tag{48}$$

The angular acceleration of the rear frame is

$$\dot{\boldsymbol{\omega}}_{\mathrm{rf}} = \begin{pmatrix} \ddot{\phi} \\ 0 \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} \ddot{\phi} \\ 0 \\ f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix}, \tag{49}$$

and the time derivative of the angular velocity of the rear wheel is

$$\dot{\boldsymbol{\omega}}_{\mathrm{rw}} = \begin{pmatrix} \ddot{\phi} \\ \ddot{\theta}_{\mathrm{rw}} \\ \ddot{\psi} \end{pmatrix} = \begin{pmatrix} \ddot{\phi} \\ \ddot{\theta}_{\mathrm{rw}} \\ f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix}. \tag{50}$$

The acceleration of the contact point in the rear frame coordinate system is

$$\ddot{\bar{\mathbf{c}}}_{\mathrm{rw}} = (\mathbf{I} - \tilde{\boldsymbol{\phi}}_{\mathrm{rf}})\ddot{\mathbf{c}}_{\mathrm{rw}} = \begin{pmatrix} 1 & \psi & 0 \\ -\psi & 1 & \phi \\ 0 & -\phi & 1 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{v} \\ vf\dot{\beta} + v^2\beta\cos\lambda/w \\ 0 \end{pmatrix}. \tag{51}$$

The acceleration of the centre of the wheel is

$$\begin{aligned} \ddot{\bar{\mathbf{x}}}_{\mathrm{rw}} &= \ddot{\bar{\mathbf{c}}}_{\mathrm{rw}} + \dot{\boldsymbol{\omega}}_{\mathrm{rf}} \times (\bar{\mathbf{x}}_{\mathrm{rw}} - \bar{\mathbf{c}}_{\mathrm{rw}}) \\ &= \begin{pmatrix} \dot{v} \\ vf\dot{\beta} + v^2\beta\cos\lambda/w \\ 0 \end{pmatrix} + \begin{pmatrix} \ddot{\phi} \\ 0 \\ f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ -R_{\mathrm{rw}} \end{pmatrix} \\ &= \begin{pmatrix} \dot{v} \\ R_{\mathrm{rw}}\ddot{\phi} + vf\dot{\beta} + v^2\beta\cos\lambda/w \\ 0 \end{pmatrix}, \end{aligned} \tag{52}$$

while the acceleration of the centre of mass of the rear frame is

$$\begin{aligned} \ddot{\bar{\mathbf{x}}}_{\mathrm{rf}} &= \ddot{\bar{\mathbf{x}}}_{\mathrm{rw}} + \dot{\boldsymbol{\omega}}_{\mathrm{rf}} \times (\bar{\mathbf{x}}_{\mathrm{rf}} - \bar{\mathbf{x}}_{\mathrm{rw}}) \\ &= \begin{pmatrix} \dot{v} \\ R_{\mathrm{rw}}\ddot{\phi} + vf\dot{\beta} + v^2\beta\cos\lambda/w \\ 0 \end{pmatrix} + \begin{pmatrix} \ddot{\phi} \\ 0 \\ f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix} \times \begin{pmatrix} x_{\mathrm{rf}} \\ 0 \\ R_{\mathrm{rw}} + z_{\mathrm{rf}} \end{pmatrix} \\ &= \begin{pmatrix} \dot{v} \\ -z_{\mathrm{rf}}\ddot{\phi} + fx_{\mathrm{rf}}\ddot{\beta} + (x_{\mathrm{rf}} + t)v\dot{\beta}\cos\lambda/w + (x_{\mathrm{rf}}\dot{v} + v^2)\beta\cos\lambda/w \\ 0 \end{pmatrix}. \end{aligned} \tag{53}$$

Likewise, the acceleration of the steering point is

$$\ddot{\mathbf{x}}_{\mathrm{sp}} = \ddot{\mathbf{x}}_{\mathrm{rw}} + \dot{\boldsymbol{\omega}}_{\mathrm{rf}} \times (\bar{\mathbf{x}}_{\mathrm{sp}} - \bar{\mathbf{x}}_{\mathrm{rw}})$$
$$= \begin{pmatrix} \dot{v} \\ R_{\mathrm{rw}}\ddot{\phi} + vf\dot{\beta} + v^2\beta\cos\lambda/w \\ 0 \end{pmatrix} + \begin{pmatrix} \ddot{\phi} \\ 0 \\ f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix} \times \begin{pmatrix} w+t \\ 0 \\ R_{\mathrm{rw}} \end{pmatrix}$$
$$= \begin{pmatrix} \dot{v} \\ f(w+t)\ddot{\beta} + (w+2t)v\dot{\beta}\cos\lambda/w + [(w+t)\dot{v} + v^2]\beta\cos\lambda/w \\ 0 \end{pmatrix}. \tag{54}$$

If this is transformed to the system of the front frame, we obtain

$$\ddot{\hat{\mathbf{x}}}_{\mathrm{sp}} = (\mathbf{I} - \beta\tilde{\boldsymbol{\lambda}})\ddot{\mathbf{x}}_{\mathrm{sp}}$$
$$= \begin{pmatrix} 1 & \beta\cos\lambda & 0 \\ -\beta\cos\lambda & 1 & \beta\sin\lambda \\ 0 & -\beta\sin\lambda & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} \dot{v} \\ f(w+t)\ddot{\beta} + (w+2t)v\dot{\beta}\cos\lambda/w + [(w+t)\dot{v} + v^2]\beta\cos\lambda/w \\ 0 \end{pmatrix} \tag{55}$$
$$= \begin{pmatrix} \dot{v} \\ f(w+t)\ddot{\beta} + (w+2t)v\dot{\beta}\cos\lambda/w + (t\dot{v} + v^2)\beta\cos\lambda/w \\ 0 \end{pmatrix}.$$

The angular acceleration of the front frame is

$$\dot{\boldsymbol{\omega}}_{\mathrm{ff}} = \begin{pmatrix} \ddot{\phi} + \ddot{\beta}\sin\lambda \\ 0 \\ (\cos\lambda + f)\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix}, \tag{56}$$

and the time derivative of the component vector of the angular velocity of the front wheel is

$$\dot{\boldsymbol{\omega}}_{\mathrm{fw}} = \begin{pmatrix} \ddot{\phi} + \ddot{\beta}\sin\lambda \\ \ddot{\theta}_{\mathrm{rw}}R_{\mathrm{rw}}/R_{\mathrm{fw}} \\ (\cos\lambda + f)\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix}. \tag{57}$$

With this, the acceleration of the centre of the front wheel becomes

$$
\begin{aligned}
\ddot{\hat{\mathbf{x}}}_{\mathrm{fw}} &= \ddot{\hat{\mathbf{x}}}_{\mathrm{sp}} + \dot{\hat{\boldsymbol{\omega}}}_{\mathrm{ff}} \times (\hat{\mathbf{x}}_{\mathrm{fw}} - \hat{\mathbf{x}}_{\mathrm{sp}}) \\
&= \begin{pmatrix} \dot{v} \\ f(w+t)\ddot{\beta} + (w+2t)v\dot{\beta}\cos\lambda/w + (t\dot{v}+v^2)\beta\cos\lambda/w \\ 0 \end{pmatrix} \\
&+ \begin{pmatrix} \ddot{\phi} + \ddot{\beta}\sin\lambda \\ 0 \\ (\cos\lambda + f)\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix} \times \begin{pmatrix} -t \\ 0 \\ -R_{\mathrm{fw}} \end{pmatrix} \\
&= \begin{pmatrix} \dot{v} \\ R_{\mathrm{fw}}\ddot{\phi} + (u_{\mathrm{fw}} + fw)\ddot{\beta} + (w+t)v\dot{\beta}\cos\lambda/w + v^2\beta\cos\lambda/w \\ 0 \end{pmatrix},
\end{aligned}
\tag{58}
$$

and, similarly, for the centre of mass of the front frame, the accelerations are

$$
\begin{aligned}
\ddot{\hat{\mathbf{x}}}_{\mathrm{ff}} &= \ddot{\hat{\mathbf{x}}}_{\mathrm{sp}} + \dot{\hat{\boldsymbol{\omega}}}_{\mathrm{ff}} \times (\hat{\mathbf{x}}_{\mathrm{ff}} - \hat{\mathbf{x}}_{\mathrm{sp}}) \\
&= \begin{pmatrix} \dot{v} \\ f(w+t)\ddot{\beta} + (w+2t)v\dot{\beta}\cos\lambda/w + (t\dot{v}+v^2)\beta\cos\lambda/w \\ 0 \end{pmatrix} \\
&+ \begin{pmatrix} \ddot{\phi} + \ddot{\beta}\sin\lambda \\ 0 \\ (\cos\lambda + f)\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w \end{pmatrix} \times \begin{pmatrix} x_{\mathrm{ff}} - w - t \\ 0 \\ z_{\mathrm{ff}} \end{pmatrix} \\
&= \begin{pmatrix} \dot{v} \\ -z_{\mathrm{ff}}\ddot{\phi} + (u_{\mathrm{ff}} + fx_{\mathrm{ff}})\ddot{\beta} + (x_{\mathrm{ff}}+t)v\dot{\beta}\cos\lambda/w + [(x_{\mathrm{ff}}-w)\dot{v} + v^2]\beta\cos\lambda/w \\ 0 \end{pmatrix}.
\end{aligned}
\tag{59}
$$

This ends the acceleration analysis.

7. Equations of motion

First, the equations of motion for the individual bodies are derived, and reaction forces and moments are retained. Then the differential equations for the system are derived with the principle of virtual work. First, the vector of the acceleration of gravity is transformed to the local frames as

$$
\bar{\mathbf{g}} = (\mathbf{I} - \tilde{\boldsymbol{\phi}}_{\mathrm{rf}})\mathbf{g} = \begin{pmatrix} 1 & \psi & 0 \\ -\psi & 1 & \phi \\ 0 & -\phi & 1 \end{pmatrix} \begin{pmatrix} g_X \\ 0 \\ g_Z \end{pmatrix} = \begin{pmatrix} g_X \\ -g_X\psi + g_Z\phi \\ g_Z \end{pmatrix}, \tag{60}
$$

$$
\hat{\mathbf{g}} = (\mathbf{I} - \tilde{\boldsymbol{\phi}}_{\mathrm{ff}})\mathbf{g} = \begin{pmatrix} 1 & \psi + \beta\cos\lambda & 0 \\ -\psi - \beta\cos\lambda & 1 & \phi + \beta\sin\lambda \\ 0 & -\phi - \beta\sin\lambda & 1 \end{pmatrix} \begin{pmatrix} g_X \\ 0 \\ g_Z \end{pmatrix}
$$

$$
= \begin{pmatrix} g_X \\ -g_X\psi + g_Z\phi + (-g_X\cos\lambda + g_Z\sin\lambda)\beta \\ g_Z \end{pmatrix}. \tag{61}
$$

The equations of motion for the rear wheel are

$$
\mathbf{A}\dot{\bar{\boldsymbol{\omega}}}_{\mathrm{rw}} + \bar{\boldsymbol{\omega}}_{\mathrm{rf}} \times (\mathbf{A}\bar{\boldsymbol{\omega}}_{\mathrm{rw}}) = \bar{\mathbf{M}}_{\mathrm{rw}}, \qquad m_{\mathrm{rw}}\ddot{\bar{\mathbf{x}}}_{\mathrm{rw}} - m_{\mathrm{rw}}\bar{\mathbf{g}} = \bar{\mathbf{F}}_{\mathrm{rw}}. \tag{62}
$$

Here, $\bar{\mathbf{M}}_{\mathrm{rw}}$ represents the resultant reaction moment with the inclusion of the breaking moment and $\bar{\mathbf{F}}_{\mathrm{rw}}$ represents the resultant reaction force. Written out, these equations become

$$
\begin{pmatrix} A_{xx}\ddot{\phi} + A_{yy}(fv\dot{\beta} + v^2\beta\cos\lambda/w)/R_{\mathrm{rw}} \\ A_{yy}\ddot{\theta}_{\mathrm{rw}} \\ A_{xx}(f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w) - A_{yy}v\dot{\phi}/R_{\mathrm{rw}} \end{pmatrix} = \begin{pmatrix} \bar{M}_{\mathrm{rw},x} \\ \bar{M}_{\mathrm{rw},y} \\ \bar{M}_{\mathrm{rw},z} \end{pmatrix} \tag{63}
$$

and

$$
\begin{pmatrix} m_{\mathrm{rw}}(\dot{v} - g_X) \\ m_{\mathrm{rw}}(R_{\mathrm{rw}}\ddot{\phi} + vf\dot{\beta} + v^2\beta\cos\lambda/w + g_X\psi - g_Z\phi) \\ -m_{\mathrm{rw}}g_Z \end{pmatrix} = \begin{pmatrix} \bar{F}_{\mathrm{rw},x} \\ \bar{F}_{\mathrm{rw},y} \\ \bar{F}_{\mathrm{rw},z} \end{pmatrix}. \tag{64}
$$

The equations of motion for the rear frame are

$$
\mathbf{B}\dot{\bar{\boldsymbol{\omega}}}_{\mathrm{rf}} = \bar{\mathbf{M}}_{\mathrm{rf}}, \qquad m_{\mathrm{rf}}\ddot{\bar{\mathbf{x}}}_{\mathrm{rf}} - m_{\mathrm{rf}}\bar{\mathbf{g}} = \bar{\mathbf{F}}_{\mathrm{rf}}. \tag{65}
$$

Written out, these equations become

$$
\begin{pmatrix} B_{xx}\ddot{\phi} + B_{xz}(f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w) \\ 0 \\ B_{xz}\ddot{\phi} + B_{zz}(f\ddot{\beta} + v\dot{\beta}\cos\lambda/w + \dot{v}\beta\cos\lambda/w) \end{pmatrix} = \begin{pmatrix} \bar{M}_{\mathrm{rf},x} \\ \bar{M}_{\mathrm{rf},y} \\ \bar{M}_{\mathrm{rf},z} \end{pmatrix} \tag{66}
$$

and

$$
\begin{pmatrix} m_{\mathrm{rf}}(\dot{v} - g_X) \\ m_{\mathrm{rf}}[-z_{\mathrm{rf}}\ddot{\phi} + fx_{\mathrm{rf}}\ddot{\beta} + (x_{\mathrm{rf}} + t)v\dot{\beta}\cos\lambda/w + (x_{\mathrm{rf}}\dot{v} + v^2)\beta\cos\lambda/w + g_X\psi - g_Z\phi] \\ -m_{\mathrm{rf}}g_Z \end{pmatrix}
$$

$$
= \begin{pmatrix} \bar{F}_{\mathrm{rf},x} \\ \bar{F}_{\mathrm{rf},y} \\ \bar{F}_{\mathrm{rf},z} \end{pmatrix} \tag{67}
$$

The equations of motion for the front frame are

$$
\mathbf{C}\dot{\hat{\boldsymbol{\omega}}}_{\mathrm{ff}} = \hat{\mathbf{M}}_{\mathrm{ff}}, \qquad m_{\mathrm{ff}}\ddot{\hat{\mathbf{x}}}_{\mathrm{ff}} - m_{\mathrm{ff}}\hat{\mathbf{g}} = \hat{\mathbf{F}}_{\mathrm{ff}}. \tag{68}
$$

Written out, these equations become

$$\begin{pmatrix} C_{xx}(\ddot\phi + \ddot\beta\sin\lambda) + C_{xz}[(\cos\lambda + f)\ddot\beta + v\dot\beta\cos\lambda/w + \dot v\beta\cos\lambda/w] \\ 0 \\ C_{xz}(\ddot\phi + \ddot\beta\sin\lambda) + C_{zz}[(\cos\lambda + f)\ddot\beta + v\dot\beta\cos\lambda/w + \dot v\beta\cos\lambda/w] \end{pmatrix} = \begin{pmatrix} \hat M_{\mathrm{ff},x} \\ \hat M_{\mathrm{ff},y} \\ \hat M_{\mathrm{ff},z} \end{pmatrix} \tag{69}$$

and

$$\begin{aligned} &\begin{pmatrix} m_{\mathrm{ff}}\dot v \\ m_{\mathrm{ff}}\{-z_{\mathrm{ff}}\ddot\phi + (u_{\mathrm{ff}} + f x_{\mathrm{ff}})\ddot\beta + (x_{\mathrm{ff}} + t)v\dot\beta\cos\lambda/w + [(x_{\mathrm{ff}} - w)\dot v + v^2]\beta\cos\lambda/w\} \\ 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} m_{\mathrm{ff}} g_X \\ m_{\mathrm{ff}}[-g_X\psi + g_Z\phi + (-g_X\cos\lambda + g_Z\sin\lambda)\beta] \\ m_{\mathrm{ff}} g_Z \end{pmatrix} = \begin{pmatrix} \hat F_{\mathrm{ff},x} \\ \hat F_{\mathrm{ff},y} \\ \hat F_{\mathrm{ff},z} \end{pmatrix}. \end{aligned} \tag{70}$$

The equations of motion for the front wheel are

$$\mathbf{D}\dot{\hat{\boldsymbol\omega}}_{\mathrm{fw}} + \hat{\boldsymbol\omega}_{\mathrm{ff}} \times (\mathbf{D}\hat{\boldsymbol\omega}_{\mathrm{fw}}) = \hat{\mathbf{M}}_{\mathrm{fw}}, \qquad m_{\mathrm{fw}}\ddot{\hat{\mathbf{x}}}_{\mathrm{fw}} - m_{\mathrm{fw}}\hat{\mathbf{g}} = \hat{\mathbf{F}}_{\mathrm{fw}}. \tag{71}$$

Written out, these equations become

$$\begin{pmatrix} D_{xx}(\ddot\phi + \ddot\beta\sin\lambda) + D_{yy}[(\cos\lambda + f)v\dot\beta + v^2\beta\cos\lambda/w]/R_{\mathrm{fw}} \\ D_{yy}\ddot\theta_{\mathrm{rw}}R_{\mathrm{rw}}/R_{\mathrm{fw}} \\ D_{xx}[(\cos\lambda + f)\ddot\beta + v\dot\beta\cos\lambda/w + \dot v\beta\cos\lambda/w] - D_{yy}v(\dot\phi + \dot\beta\sin\lambda)/R_{\mathrm{fw}} \end{pmatrix} = \begin{pmatrix} \hat M_{\mathrm{fw},x} \\ \hat M_{\mathrm{fw},y} \\ \hat M_{\mathrm{fw},z} \end{pmatrix} \tag{72}$$

and

$$\begin{aligned} &\begin{pmatrix} m_{\mathrm{fw}}\dot v \\ m_{\mathrm{fw}}[R_{\mathrm{fw}}\ddot\phi + (u_{\mathrm{fw}} + f w)\ddot\beta + (w + t)v\dot\beta\cos\lambda/w + v^2\beta\cos\lambda/w] \\ 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} m_{\mathrm{fw}} g_X \\ m_{\mathrm{fw}}[-g_X\psi + g_Z\phi + (-g_X\cos\lambda + g_Z\sin\lambda)\beta] \\ m_{\mathrm{fw}} g_Z \end{pmatrix} = \begin{pmatrix} \hat F_{\mathrm{fw},x} \\ \hat F_{\mathrm{fw},y} \\ \hat F_{\mathrm{fw},z} \end{pmatrix} \end{aligned} \tag{73}$$

The virtual work done by the braking moments is

$$M_{\mathrm{rw}}\delta\theta_{\mathrm{rw}} + M_{\mathrm{fw}}\delta\theta_{\mathrm{fw}} = (M_{\mathrm{rw}} + M_{\mathrm{fw}}R_{\mathrm{rw}}/R_{\mathbf{fw}})\delta\theta_{\mathrm{rw}} - (M_{\mathrm{fw}}/R_{\mathrm{fw}})f(w + t)\beta\delta\beta\cos\lambda. \tag{74}$$

The equations of motion are found by eliminating the reaction forces. This is most easily done by taking the inner product of each of the eight equations with the corresponding virtual displacements or rotations. There are 21 independent reaction forces and three equations of motion, for which we have 24 equations, so in the general case, the system can be solved. If we take first variations with respect to the rotation of the rear wheel, we recover the equation of motion for the trivial solution as given in Eq. (4). If we collect all terms corresponding to virtual variations of the roll angle, we obtain the equation of motion

$$
\begin{aligned}
&[A_{xx} + m_{\mathrm{rw}} R_{\mathrm{rw}}^2 + B_{xx} + m_{\mathrm{rf}} z_{\mathrm{rf}}^2 + C_{xx} + m_{\mathrm{ff}} z_{\mathrm{ff}}^2 + D_{xx} + m_{\mathrm{fw}} R_{\mathrm{fw}}^2]\ddot{\phi} \\
&\quad +[f B_{xz} - f m_{\mathrm{rf}} x_{\mathrm{rf}} z_{\mathrm{rf}} + C_{xx} \sin\lambda + C_{xz}(\cos\lambda + f) - m_{\mathrm{ff}}(u_{\mathrm{ff}} + f x_{\mathrm{ff}}) z_{\mathrm{ff}} + D_{xx} \sin\lambda \\
&\qquad + m_{\mathrm{fw}}(u_{\mathrm{fw}} + f w) R_{\mathrm{fw}}]\ddot{\beta} \\
&\quad +[f A_{yy}/R_{\mathrm{rw}} + f m_{\mathrm{rw}} R_{\mathrm{rw}} + B_{xz} \cos\lambda/w - m_{\mathrm{rf}}(x_{\mathrm{rf}} + t) z_{\mathrm{rf}} \cos\lambda/w + C_{xz} \cos\lambda/w \\
&\qquad - m_{\mathrm{ff}}(x_{\mathrm{ff}} + t) z_{\mathrm{ff}} \cos\lambda/w + (D_{yy}/R_{\mathrm{fw}})(\cos\lambda + f) + m_{\mathrm{fw}}(w + t) R_{\mathrm{fw}} \cos\lambda/w] v\dot{\beta} \\
&\quad +[-m_{\mathrm{rw}} R_{\mathrm{rw}} + m_{\mathrm{rf}} z_{\mathrm{rf}} + m_{\mathrm{ff}} z_{\mathrm{ff}} - m_{\mathrm{fw}} R_{\mathrm{fw}}](g_Z \phi - g_X \psi) \\
&\quad +[-f m_{\mathrm{rw}} R_{\mathrm{rw}} + f m_{\mathrm{rf}} z_{\mathrm{rf}} + f m_{\mathrm{ff}} z_{\mathrm{ff}} - f m_{\mathrm{fw}} R_{\mathrm{fw}}] g_X \beta \\
&\quad +\{-f m_{\mathrm{rf}} x_{\mathrm{rf}} + m_{\mathrm{ff}} \sin\lambda z_{\mathrm{ff}} - m_{\mathrm{ff}}[(x_{\mathrm{ff}} - w - t)\cos\lambda + f x_{\mathrm{ff}}] - m_{\mathrm{fw}} R_{\mathrm{fw}} \sin\lambda\} g_Z \beta \\
&\quad +[A_{yy}/R_{\mathrm{rw}} + m_{\mathrm{rw}} R_{\mathrm{rw}} - m_{\mathrm{rf}} z_{\mathrm{rf}} - m_{\mathrm{ff}} z_{\mathrm{ff}} + D_{yy}/R_{\mathrm{fw}} + m_{\mathrm{fw}} R_{\mathrm{fw}}] v^2 \beta \cos\lambda/w \\
&\quad +[t A_{yy}/R_{\mathrm{rw}} + t m_{\mathrm{rw}} R_{\mathrm{rw}} + B_{xz} - m_{\mathrm{rf}}(x_{\mathrm{rf}} + t) z_{\mathrm{rf}} + C_{xz} - m_{\mathrm{ff}}(x_{\mathrm{ff}} + t) z_{\mathrm{ff}} \\
&\qquad +(w + t) D_{yy}/R_{\mathrm{fw}} + m_{\mathrm{fw}} R_{\mathrm{fw}}(w + t)]\dot{v}\beta \cos\lambda/w = 0.
\end{aligned} \tag{75}
$$

For the steering angle β:

$$
\begin{aligned}
&[f B_{xz} - f m_{\mathrm{rf}} x_{\mathrm{rf}} z_{\mathrm{rf}} + C_{xx} \sin\lambda + C_{xz}(\cos\lambda + f) - m_{\mathrm{ff}}(u_{\mathrm{ff}} + f x_{\mathrm{ff}}) z_{\mathrm{ff}} + D_{xx} \sin\lambda \\
&\qquad + m_{\mathrm{fw}}(u_{\mathrm{fw}} + f w) R_{\mathrm{fw}}]\ddot{\phi} \\
&\quad +[f^2 A_{xx} + f^2 B_{zz} + f^2 m_{\mathrm{rf}} x_{\mathrm{rf}}^2 + C_{xx} \sin^2\lambda + 2C_{xz}(\cos\lambda + f)\sin\lambda + C_{zz}(\cos\lambda + f)^2 \\
&\qquad + m_{\mathrm{ff}}(u_{\mathrm{ff}} + f x_{\mathrm{ff}})^2 + D_{xx} \sin^2\lambda + D_{xx}(\cos\lambda + f)^2 + m_{\mathrm{fw}}(u_{\mathrm{fw}} + f w)^2]\ddot{\beta} \\
&\quad +[-f A_{yy}/R_{\mathrm{rw}} - (D_{yy}/R_{\mathrm{fw}})(\cos\lambda + f)] v\dot{\phi} \\
&\quad +[f A_{xx} + f B_{zz} + f m_{\mathrm{rf}} x_{\mathrm{rf}}(x_{\mathrm{rf}} + t) + C_{xz} \sin\lambda + C_{zz}(\cos\lambda + f) + m_{\mathrm{ff}}(u_{\mathrm{ff}} + f x_{\mathrm{ff}})(x_{\mathrm{ff}} + t) \\
&\qquad + D_{xx}(\cos\lambda + f) + m_{\mathrm{fw}}(w + t)(u_{\mathrm{fw}} + f w)] v\dot{\beta} \cos\lambda/w \\
&\quad +[-f m_{\mathrm{rf}} x_{\mathrm{rf}} - m_{\mathrm{ff}}(u_{\mathrm{ff}} + f x_{\mathrm{ff}}) - m_{\mathrm{fw}}(u_{\mathrm{fw}} + f w)](g_Z \phi - g_X \psi) \\
&\quad +[-f m_{\mathrm{rw}} R_{\mathrm{rw}} \sin\lambda + f m_{\mathrm{rf}} z_{\mathrm{rf}} \sin\lambda - f m_{\mathrm{ff}}(w + t)\cos\lambda + m_{\mathrm{ff}} \cos\lambda(u_{\mathrm{ff}} + f x_{\mathrm{ff}}) \\
&\qquad - f m_{\mathrm{fw}}(w + t)\cos\lambda + m_{\mathrm{fw}} \cos\lambda(u_{\mathrm{fw}} + f w)] g_X \beta \\
&\quad +[-f m_{\mathrm{rf}} x_{\mathrm{rf}} \sin\lambda - m_{\mathrm{ff}} \sin\lambda(u_{\mathrm{ff}} + f x_{\mathrm{ff}}) - m_{\mathrm{fw}} \sin\lambda(u_{\mathrm{fw}} + f w)] g_Z \beta \\
&\quad +(M_{\mathrm{fw}}/R_{\mathrm{fw}}) f(w + t)\beta \cos\lambda \\
&\quad +[f m_{\mathrm{rf}} x_{\mathrm{rf}} + m_{\mathrm{ff}}(u_{\mathrm{ff}} + f x_{\mathrm{ff}}) + (D_{yy}/R_{\mathrm{fw}})\sin\lambda + m_{\mathrm{fw}}(u_{\mathrm{fw}} + f w)] v^2 \beta \cos\lambda/w \\
&\quad +[t \sin\lambda A_{yy}/R_{\mathrm{rw}} + f A_{xx} + t m_{\mathrm{rw}} R_{\mathrm{rw}} \sin\lambda + f B_{zz} - t m_{\mathrm{rf}} z_{\mathrm{rf}} \sin\lambda + f m_{\mathrm{rf}} x_{\mathrm{rf}}^2 + C_{xz} \sin\lambda \\
&\qquad + C_{zz}(\cos\lambda + f) + m_{\mathrm{ff}}(u_{\mathrm{ff}} + f x_{\mathrm{ff}})(x_{\mathrm{ff}} - w) + t m_{\mathrm{ff}}(w + t)\cos\lambda + D_{xx}(\cos\lambda + f) \\
&\qquad + f w(w + t) D_{yy}/R_{\mathrm{fw}}^2 + t m_{\mathrm{fw}}(w + t)\cos\lambda]\dot{v}\beta \cos\lambda/w = 0.
\end{aligned} \tag{76}
$$

These expressions can be simplified by introducing some auxiliary quantities. The moments of inertia of the complete system with respect to the rear wheel contact point:

$$
\begin{aligned}
T_{xx} &= A_{xx} + B_{xx} + C_{xx} + D_{xx} + m_{\mathrm{rw}} R_{\mathrm{rw}}^2 + m_{\mathrm{rf}} z_{\mathrm{rf}}^2 + m_{\mathrm{ff}} z_{\mathrm{ff}}^2 + m_{\mathrm{fw}} R_{\mathrm{fw}}^2, \\
T_{xz} &= B_{xz} + C_{xz} - m_{\mathrm{rf}} x_{\mathrm{rf}} z_{\mathrm{rf}} - m_{\mathrm{ff}} x_{\mathrm{ff}} z_{\mathrm{ff}} + m_{\mathrm{fw}} w R_{\mathrm{fw}}, \\
T_{zz} &= A_{xx} + B_{zz} + C_{zz} + D_{xx} + m_{\mathrm{rf}} x_{\mathrm{rf}}^2 + m_{\mathrm{ff}} x_{ff}^2 + m_{\mathrm{fw}} w^2.
\end{aligned} \tag{77}
$$

Similarly, the skew moments of inertia of the front frame plus front wheel are defined as

$$
\begin{aligned}
F_{\lambda\lambda} &= C_{xx} \sin^2\lambda + C_{zz} \cos^2\lambda + 2C_{xz} \sin\lambda \cos\lambda + D_{xx} + m_{\mathrm{ff}} u_{\mathrm{ff}}^2 + m_{\mathrm{fw}} u_{\mathrm{fw}}^2, \\
F_{\lambda x} &= C_{xx} \sin\lambda + C_{xz} \cos\lambda + D_{xx} \sin\lambda - m_{\mathrm{ff}} u_{\mathrm{ff}} z_{\mathrm{ff}} + m_{\mathrm{fw}} R_{\mathrm{fw}} u_{\mathrm{fw}} \\
F_{\lambda z} &= C_{xz} \sin\lambda + C_{zz} \cos\lambda + D_{xx} \cos\lambda + m_{\mathrm{ff}} x_{\mathrm{ff}} u_{\mathrm{ff}} + m_{\mathrm{fw}} w u_{\mathrm{fw}}.
\end{aligned} \tag{78}
$$

A number of static moments are introduced as

$$
\begin{aligned}
&S_x = m_\mathrm{t} z_\mathrm{t} = -m_\mathrm{rw} R_\mathrm{rw} + m_\mathrm{rf} z_\mathrm{rf} + m_\mathrm{ff} z_\mathrm{ff} - m_\mathrm{fw} R_\mathrm{fw}\\
&S_z = m_\mathrm{t} x_\mathrm{t} = m_\mathrm{rf} x_\mathrm{rf} + m_\mathrm{ff} x_\mathrm{ff} + m_\mathrm{fw} w\\
&S_\mathrm{rw} = A_{yy}/R_\mathrm{rw},\\
&S_\mathrm{fw} = D_{yy}/R_\mathrm{fw},\\
&S_\mathrm{w} = S_\mathrm{rw} + S_\mathrm{fw}\\
&s_\lambda = m_\mathrm{ff} u_\mathrm{ff} + m_\mathrm{fw} u_\mathrm{fw}
\end{aligned}
\tag{79}
$$

In this way, the two equations of motion simplify to

$$
\begin{aligned}
&T_{xx}\ddot{\phi} + (F_{\lambda x} + fT_{xz})\ddot{\beta} + (fS_\mathrm{w} + S_\mathrm{fw}\cos\lambda - fS_x + T_{xz}\cos\lambda/w)(v\dot{\beta} + \dot{v}\beta)\\
&\quad +(S_\mathrm{w} - S_x)v^2\beta\cos\lambda/w + S_x[g_Z\phi - g_X(\psi - f\beta)] - (fS_z + s_\lambda)g_Z\beta = 0,
\end{aligned}
\tag{80}
$$

and

$$
\begin{aligned}
&(F_{\lambda x} + tT_{xz})\ddot{\phi} + (F_{\lambda\lambda} + 2fF_{\lambda z} + f^2T_{zz})\ddot{\beta} + (-fS_\mathrm{w} - S_\mathrm{fw}\cos\lambda)v\dot{\phi}\\
&\quad +[(F_{\lambda z} + fT_{zz})\cos\lambda/w + f^2S_z + fs_\lambda]v\dot{\beta} + (s_\lambda + fS_z)(-g_Z\phi + g_X\psi)\\
&\quad +(-fS_z\sin\lambda - s_\lambda\sin\lambda)g_Z\beta + [f(S_x\sin\lambda + s_\lambda) + s_\lambda\cos\lambda]g_X\beta\\
&\quad +[(D_{yy}/R_\mathrm{fw}^2)\dot{v} + (M_\mathrm{fw}/R_\mathrm{fw})]f(w+t)\beta\cos\lambda\\
&\quad +(fS_z + s_\lambda + S_\mathrm{fw}\sin\lambda)v^2\beta\cos\lambda/w\\
&\quad +[(F_{\lambda z} + fT_{zz})\cos\lambda/w + fS_\mathrm{rw}\sin\lambda - fS_x\sin\lambda - (\cos\lambda + f)s_\lambda]\dot{v}\beta = 0.
\end{aligned}
\tag{81}
$$

These equations can be written in the standard matrix-vector form

$$
\begin{aligned}
\mathbf{M}\ddot{\mathbf{q}}^\mathrm{d} + \mathbf{C}\dot{\mathbf{q}}^\mathrm{d} + \mathbf{K}^d\mathbf{q}^\mathrm{d} + \mathbf{K}^k\mathbf{q}^\mathrm{k} = \mathbf{0},\\
\dot{\mathbf{q}}^\mathrm{k} = \mathbf{A}\dot{\mathbf{q}}^\mathrm{d} + \mathbf{B}^\mathrm{d}\mathbf{q}^\mathrm{d} + \mathbf{B}^\mathrm{k}\mathbf{q}^\mathrm{k}.
\end{aligned}
\tag{82}
$$

Here, $\mathbf{q}^\mathrm{d} = (\phi, \beta)^T$ is the vector of the dynamic degrees of freedom, $\mathbf{q}^\mathrm{k} = (\psi)$ is the vector of the kinematic coordinates and $\mathbf{M}$ is the mass matrix. The damping matrix $\mathbf{C}$, the stiffness matrices $\mathbf{K}^\mathrm{d}$ and $\mathbf{K}^\mathrm{k}$, and the kinematic matrices $\mathbf{A}$, $\mathbf{B}^\mathrm{d}$ and $\mathbf{B}^\mathrm{k}$ have the structure

$$
\begin{aligned}
&\mathbf{C} = {}^1\mathbf{C}v,\\
&\mathbf{K}^\mathrm{d} = {}^0\mathbf{K}^\mathrm{d} + {}^1\mathbf{K}^\mathrm{d}\dot{v} + {}^2\mathbf{K}^\mathrm{d}v^2,\\
&\mathbf{K}^\mathrm{k} = \mathbf{K}^\mathrm{k},\\
&\mathbf{A} = \mathbf{A},\\
&\mathbf{B}^\mathrm{d} = {}^1\mathbf{B}^\mathrm{d}v,\\
&\mathbf{B}^\mathrm{k} = \mathbf{0}.
\end{aligned}
\tag{83}
$$

The components of the constituent matrices, which are independent of v and $\dot{v}$, are

$$
\begin{aligned}
&M_{11} = T_{xx},\\
&M_{12} = M_{21} = F_{\lambda x} + fT_{xz},\\
&M_{22} = F_{\lambda\lambda} + 2fF_{\lambda z} + f^2 T_{zz},\\
&{}^1C_{11} = 0,\\
&{}^1C_{12} = fS_{\mathrm{w}} + S_{\mathrm{fw}}\cos\lambda - fS_x + T_{xz}\cos\lambda/w,\\
&{}^1C_{21} = -fS_{\mathrm{w}} - S_{\mathrm{fw}}\cos\lambda,\\
&{}^1C_{22} = (F_{\lambda z} + fT_{zz})\cos\lambda/w + f^2 S_z + fs_\lambda,\\
&{}^0K^{\mathrm{d}}_{11} = S_x g_Z,\\
&{}^0K^{\mathrm{d}}_{12} = fS_x g_X - (fS_z + s_\lambda)g_Z,\\
&{}^0K^{\mathrm{d}}_{21} = -(s_\lambda + fS_z)g_Z,\\
&{}^0K^{\mathrm{d}}_{22} = (-fS_z\sin\lambda - s_\lambda\sin\lambda)g_Z + [f(S_x\sin\lambda + s_\lambda) + s_\lambda\cos\lambda]g_X\\
&\qquad +(M_{\mathrm{fw}}/R_{\mathrm{fw}})f(w+t)\cos\lambda,\\
&{}^1K^{\mathrm{d}}_{11} = 0,\\
&{}^1K^{\mathrm{d}}_{12} = fS_{\mathrm{w}} + S_{\mathrm{fw}}\cos\lambda - fS_x + T_{xz}\cos\lambda/w,\\
&{}^1K^{\mathrm{d}}_{21} = 0,\\
&{}^1K^{\mathrm{d}}_{22} = (D_{yy}/R^2_{\mathrm{fw}})f(w+t)\cos\lambda\\
&\qquad +(F_{\lambda z} + fT_{zz})\cos\lambda/w + fS_{\mathrm{rw}}\sin\lambda - fS_x\sin\lambda - (\cos\lambda + f)s_\lambda,\\
&{}^2K^{\mathrm{d}}_{11} = 0,\\
&{}^2K^{\mathrm{d}}_{12} = (S_{\mathrm{w}} - S_x)\cos\lambda/w,\\
&{}^2K^{\mathrm{d}}_{21} = 0,\\
&{}^2K^{\mathrm{d}}_{22} = (fS_z + s_\lambda + S_{\mathrm{fw}}\sin\lambda)\cos\lambda/w,\\
&K^{\mathrm{k}}_{11} = -S_x g_X,\\
&K^{\mathrm{k}}_{21} = (s_\lambda + fS_z)g_X,\\
&A_{11} = 0,\\
&A_{12} = f,\\
&{}^1B^{\mathrm{d}}_{11} = 0,\\
&{}^1B^{\mathrm{d}}_{12} = \cos\lambda/w.
\end{aligned}
\tag{84}
$$

It is seen that the system, if it is re-written as a first-order system, is of order five and that there is an eigenvalue zero. It is interesting to see that even for zero velocity and acceleration, the stiffness matrix does not show any apparent symmetry.

References

[1] Whipple, '...', 1899.

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