Neĭmark and Fufaev, 1967

In 1967 Neĭmark and Fufaev (N & F) with a brief reference to Döhring [1955] derived equations of motion of the bicycle as a classical example of a nonholonomic system. In their derivation they use Lagrange's equations with nonholonomic constraints for the path of the wheels and obtain the linearized equations of motion for the Basic bicycle model. It is their derivation that our Chapter III mainly follows.

The equations in their book which represent the relations between the auxiliary variables and generalized coordinates, linearized rolling constraints, kinetic energy for the rear and front part of the bicycle, potential energy of the bicycle, and equations of motion, can be found starting on p. 334 as eq. (2.10); eq. (2.15), eq. (2.26) and eq. (2.29), eq. (2.30); and eq. (2.37-38), respectively.

As mentioned in Chapter III an error is made in their formula for potential energy eq. (2.30). (The correct potential energy to second order is found in section 4 our of Appendix A.) This error results in the incorrect coefficients a_4, b_3, and b_4 in eqs. (2.37-38), where gm_2d should be replaced by $g\left(m_2d + \frac{c_1}{c}l\right)$. In addition to these corrections the reader should note that a typographical error occurs in the b_2 term on p. 344 of their text (where $\frac{1}{2}$ should read $\frac{1}{c}$) and in several other terms in the description of the geometry and viscous damping expressions. Also, in deriving the nonlinear equations they present nonlinear kinematic equations which are actually incorrect because they neglect the rise and fall (pitch) of the bicycle due to variations in the steer angle. (The linearized versions of these equations are correct as shown in section 1 of our Appendix A, however, quadratic order terms are needed to derive the correct potential energy.) By eliminating the effects of viscous damping in the steering column, and making the above corrections N & F's final equations of motion can be brought into agreement with those derived in Chapter III.

N & F refer to Döhring[4] and state that their equations agree in form, but it is unlikely they meant term for term as we have found them to be in disagreement. They also refer to a Russian book by Loǐcjanskiǐ and Lurk [1935] when analyzing a simplified model of an uncontrolled bicycle on p. 355. Because this reference was not available, it is not known if agreement actually exists, however it seems probable because N & F's equations become correct when simplified in this way. N & F do not mention any other bicycle-related works, although their massive reference list includes Carvallo [1901].

Singh and Goel, 1971

In January 1971 Singh and Goel (S & G) add steer damping to the Basic bicycle model in analyzing a Rajdoot motor scooter. In their analysis they claim to use Döhring's [1955] linearized equations of motion (which we have found to be correct) with a steering torque proportional to the the time derivative of the steer angle (viscous damping). We have not rigorously compared term by term but casual observation shows that the equations are in the same format as those of Dohring [1955].

[4] See p. 361 of their text.