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TECHNICAL REPORT

AN EVALUATION OF THE PERFORMANCE AND HANDLING QUALITIES OF BICYCLES

By: Roy S. Rice and R. Douglas Roland, Jr.

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FOREWORD

The work reported herein was performed by the Vehicle Research Department of the Cornell Aeronautical Laboratory under Contract No. 70-201 for the National Commission on Product Safety. The dynamical analysis and development of the mathematical model reported in the appendix was done under CAL internal support. The period of performance was from 20 October 1969 to 28 February 1970. Technical direction for the program was provided by Mr. Carl W. Blechschmidt of the NCPS whose patient understanding of the problems of testing bicycles in mid-winter is gratefully acknowledged.

The authors wish to thank the many young volunteer riders whose efforts during the test program are the basis for the test results given here. Our gratitude is also extended to Messrs. A. Pulley and R. Sweet of the Vehicle Research Department for the many measurements of the physical characteristics of the bicycles which they performed.

ABSTRACT

The purpose of the study described herein was to identify and measure (1) the critical parameters of bicycle design associated with their motions in the vertical longitudinal plane (braking and pitchover motions) and (2) their handling qualities. The approach involved both analytical and experimental procedures with major emphasis on the development of a mathematical model of the bicycle for use as a tool to evaluate the effects of design parameter variations on performance. The experimental studies consisted of performing a number of maneuvers (straight braking, steady state cornering, hands-off path following, serpentine tracking) on two bicycles of different basic design. Several different riders, providing a wide range of weight, were used in these tests. It was concluded that many factors interact to produce the stability and control characteristics of a given design but that front wheel brakes and short wheelbases can be singled out as having specific hazard potential. ( )

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1.0 INTRODUCTION

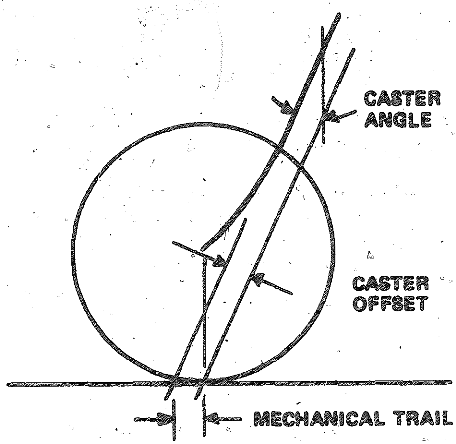
The bicycle, as a means of transportation and a source of fun, is used by millions of boys and girls throughout the country. And, certainly, a great many adults still enjoy it as a convenient form of pleasure and exercise. Almost all of these users take for granted their ability to ride any configuration of this fascinating vehicle, little recognizing the factors which make it possible to operate a device which is statically (i.e., at zero speed) unstable. The learner and novice rider, however, can attest to a need for developing a certain level of skill in order to convert this device into a practical vehicle.

In a very general way, stability and maneuverability are incompatible features in all vehicles. This means that the design is usually a tradeoff among various factors in order to achieve acceptable performance over the complete operating range of the vehicle. Emphasis is placed on those factors which best serve the purpose of the design. In the bicycle, some of these features are:

Wheelbase - All other factors being equal, increased wheelbase leads to increased stability in path-following. This is a desirable feature for straight riding and during learning, but it is detrimental to fast cornering and avoidance maneuvering.

Wheel size - For a given rotational speed, the larger wheel, because of both mass and mass distribution effects, will provide a larger gyroscopic action. This action is stabilizing (i.e., it contributes to keeping the bicycle upright) and, therefore, larger wheels are advantageous from this standpoint. The larger wheel is also better in curb climbing.

Front wheel mechanical trail - The selection of trail and caster angle (see the sketch below for definition) provide the designer with the primary means for setting the inherent stability and controllability of the bicycle. Tradeoffs involve the direction and the amount of tire self-aligning torque, gravitational and carbering effects, and desired level of steering forces. It was to enable the investigation of these interactions on stability that the mathematical model development was proposed.



2.0 SUMMARY

1. The bicycle may be likened to the automobile - the sportier the design, the more skill likely to be required for its safe operation.

2. Shorter wheelbase and smaller wheel size are detrimental to both lateral and vertical plane stability.

3. Protuberances (such as high handlebars, gear shift levers mounted on the horizontal frame member, and seat backs) which can be bumped or which inhibit the ability of the rider to get free of the bike are potential safety hazards.

4. Bicycles equipped with front wheel brakes can be stopped more quickly (that is, in shorter distances) than a similar bicycle equipped with coaster brakes. However, in some situations, such as when the rider stands upright on the pedals, hard front wheel braking can lead to forward pitchover. When this occurs, the motion is so rapid that there is little which an unsuspecting rider, even a very experienced one, can do to avoid a bad fall. The counterpart to this in hard rear-wheel braking -- a lateral breakaway -- is much more easily handled.

5. In the tests which were performed to obtain quantitative measurements of handling qualities (and these included steady state cornering and serpentine path following), the high rise bicycle did not prove to be more maneuverable at moderate speeds (10-15 mph) than the conventional bike. This is not to say that all maneuvers can be performed equally well with each design; however, it does suggest that the high riser outperforms the conventional bike only in acrobatics and in situations where its shorter overall length is essential to success. Without gearing, the high riser is not a good design for transportation; it is just too tiring to pedal at speed for distance compared to the conventional design. In essence, it is a bike to have fun with and, by incorporating

features which makes this possible, it requires somewhat more skill for its operation.

6. As part of this program, a mathematical model of the bicycle with eight degrees of motion freedom, including the three translations and the three rotations of the whole system has been developed. The mathematical model provides a capability for the evaluation of bicycle designs and the investigation of the effects of a wide range of design factors on performance. Such features include wheel size, fork angle, wheelbase, total weight and weight distribution, height of center of gravity, and tire characteristics. In this short study, it has not been possible to do more than get the simulation working properly, but it is strongly recommended that it be used to study these effects in order to achieve a better understanding of the fundamentals of bicycle stability and control.

3.0 DISCUSSION

3.1 PHYSICAL CHARACTERISTICS

The bicycles which were used in the tests are shown in Figures 1 and 2. They are considered to be representative of the two most popular bicycle styles - the high riser and the lightweight conventional design. A summary of the physical characteristics of the two bicycles is given in Table I.

TABLE I: PHYSICAL CHARACTERISTICS

	High Rise	Standard
Weight - Total	39 lbs.	39 lbs.
Front	18 lbs.	19 lbs.
Rear	21 lbs.	20 lbs.
Wheel Weight	4 lbs.	5 lbs.
Wheelbase	35 in.	43 in.
Wheel size	21 in. (OD)	26-1/2 in. (OD)
Seat Height	31-33 in.	33 in.
Caster Angle	20.5°	23°
Mechanical Trail	1-1/2 ± 1/8 in.	2-7/8 ± 1/8 in.
Max. Roll Angle*	25°	28.5°

The bicycles are standard in every way except for the modification which provides both coaster brakes and caliper brakes on each unit. The high rise bicycle was equipped with a single-speed drive only, as was the conventional model.

Additional physical measurements of several rider-bicycle combinations to obtain data on center of gravity locations were made. The results of these measurements which are defined in Figure 3 are shown in Table II.

*Measured with the pedals in a vertical plane.

Figure 1: LIGHTWEIGHT CONVENTIONAL BICYCLE

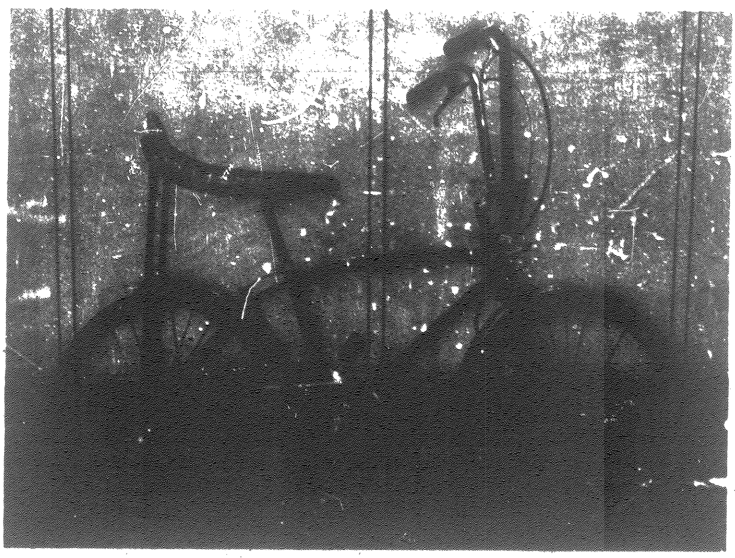
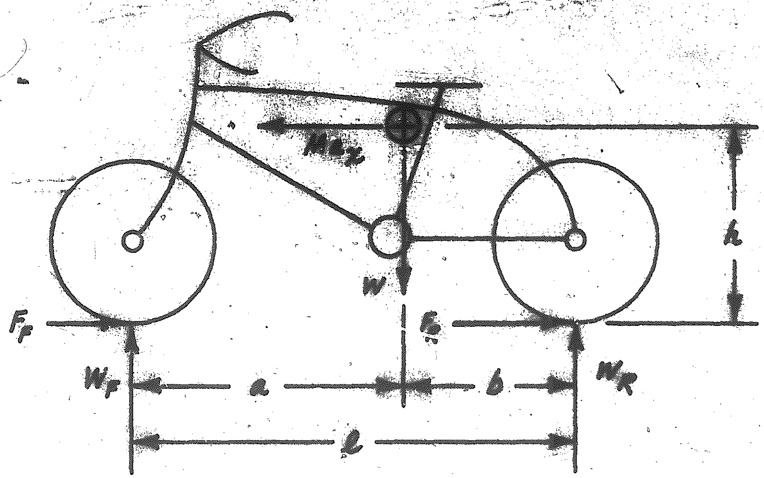


Figure 2: HIGH RISER BICYCLE



WHERE W = TOTAL WEIGHT OF RIDER AND VEHICLE
W_F = NORMAL LOAD ON FRONT WHEEL
W_R = NORMAL LOAD ON REAR WHEEL
F_F = BRAKING FORCE AT FRONT WHEEL
F_R = BRAKING FORCE AT REAR WHEEL
Ma_x = BRAKING REACTION (MASS TIMES DECELERATION)
a = LOCATION OF c.g. AFT OF FRONT WHEEL
b = LOCATION OF c.g. FORWARD OF REAR WHEEL
h = LOCATION OF c.g. ABOVE GROUND
ℓ = WHEELBASE = a + b

Figure 3 BICYCLE GEOMETRY

TABLE II: CENTER-OF-GRAVITY LOCATIONS

	High Rise	Conventional
Bicycle alone Horizontal (from front axle) Vertical (above ground) (Measured)	18-1/2 in. 18-1/2 in.	22 in. 19-3/4 in.
Bicycle and 190 lb. rider Horizontal Vertical (Measured)	26-1/2 in. 33 in.	28-3/4 in. 34 in.
Bicycle and 100 lb. rider Horizontal Vertical (Computed)	24-1/2 in. 28 in.	25-1/2 in. 30 in.

The main conclusions to be reached from these data are:

1. The center-of-gravity on the hi-riser is a larger percentage of the wheelbase aft of the front wheel than on the conventional unit. This is preferred from the standpoint of braking and downhill vertical plane stability; it is detrimental to uphill vertical plane stability. Note that, because of the shorter wheelbase, the absolute value of "a" is less for the high rise than the conventional bicycle even though the ratio of "a" to "l" is larger.

   On the high rise bicycle, the values of a/l for the rider-bicycle combinations ranged from about .70 for low weight riders to about .76 for heavy riders. Values for the standard bicycles ranged from about .60 for low weight riders to about .67 for heavy riders.

2. The vertical center-of-gravity for the "bicycle only" condition is lower for the high-rise bicycle than for the conventional unit. Low centers-of-gravity are desirable for stability (both in roll and in pitch).

These slightly lower centers-of-gravity also carry over to the rider-bicycle combinations.

3. The ratio of "a" to "h" (which should be large to avoid front pitchover in braking or curb-climbing) is about the same for both bicycles with riders. The value varies with rider weight and size and with his position and attitude on the seat.

The primary measurement made in many of the tests was forward speed of the bicycle. For this purpose, each bicycle was equipped with a manufacturer-approved speedometer which was later calibrated at three different speeds by CAL. The results of this calibration are shown in Table III.

TABLE III: BICYCLE SPEEDOMETER CALIBRATIONS

Actual Speed (MPH)	Indicated Speed (MPH)	
	Conventional	High Rise
10	7.5	9
15	13	15.5
20	19.5	21

Tests specifically aimed at determining maximum attainable speed were not performed. It was found in the course of other testing that both bicycles could be ridden at speeds in excess of 20 MPH for short distances by most of the test subjects. Such speeds could not be sustained for any period with either bicycle. It was generally agreed, as is to be expected because of its smaller wheels, that the high-rise bicycle was more tiring than the conventional unit at speeds above 15 MPH. With respect to maximum speed, it was concluded that both bicycles could be operated at high enough speeds to assure adequate testing of the front pitchover problem.

3.2 TEST PROGRAM

The test program was designed to provide general evaluations of the following characteristics:

- Type of braking system
- Size and weight of riders
- Maneuverability
- Stability
- Speed capability

The results of these tests are described briefly below.

Braking Performance

Figures 4 through 6 show stopping distances measured in a large number of test runs. In all cases, the rider was instructed to make a hard stop after attaining a given initial speed. For clarity, only the average value for several runs at a specific condition is shown. Figure 4 is a plot of stopping distance vs. initial speed for a 160 pound rider. The curves show the advantage of front wheel braking over rear wheel braking in stopping distance. Additional tests showed coaster brakes to be about equivalent to rear caliper brakes in stopping. This is shown in Figure 5 which contains data for a wide range of rider weight.

Figure 6 shows the effect of rider weight on stopping distance with caliper brakes. For these runs, a steady state speed of 15 MPH was achieved and a hard braking stop was then attempted. The data show that stopping distance increases with rider weight. This is probably due to saturation of the available brake torque with caliper brakes; no such variation was found with coaster brakes. Illustrated points are average values.

In the standup position, the rider's center-of-gravity is along a vertical line through the crank axis as shown in the following sketch. For the two bicycles under test, the dimensions A shown in the sketch are 25.5 inches and 21 inches for the conventional and high rise bicycles, respectively. The restoring moment for the shorter wheelbase bicycle is therefore some 20%

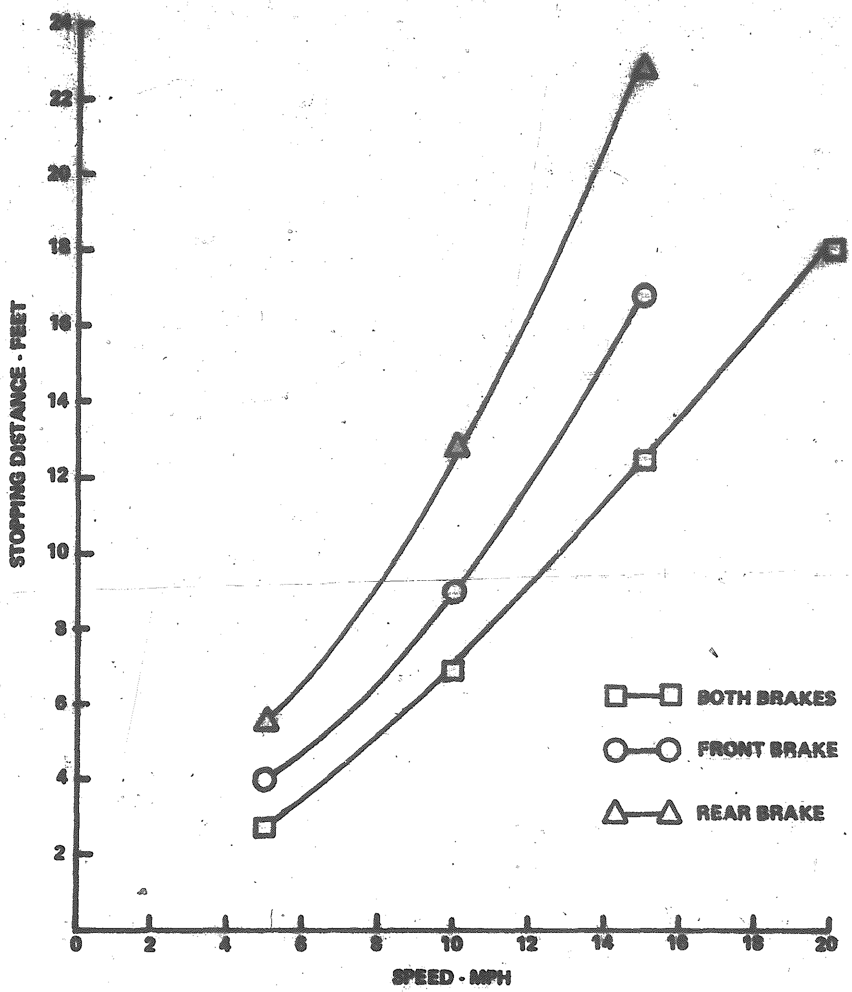


Figure 4 **BRAKING CHARACTERISTICS OF CONVENTIONAL BICYCLE WITH CALIPER BRAKES**

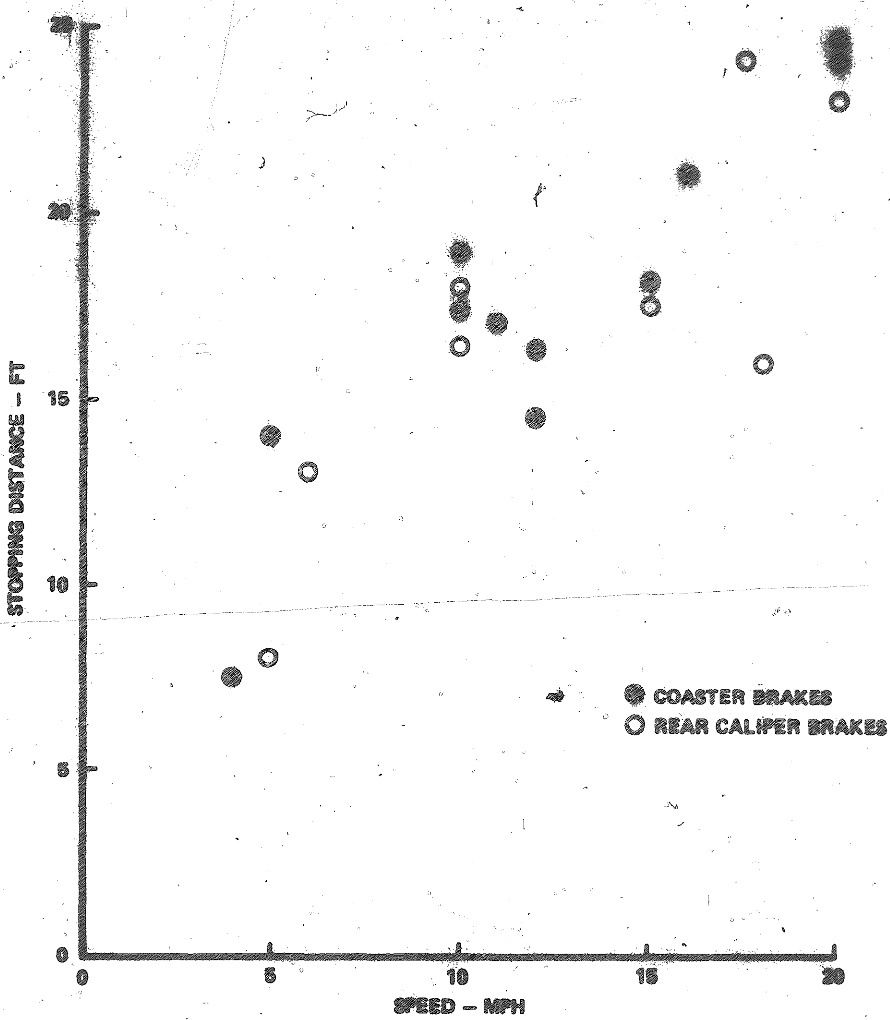


Figure 5 COMPARISON OF COASTER BRAKES AND REAR CALIPER BRAKES – CONVENTIONAL BICYCLE

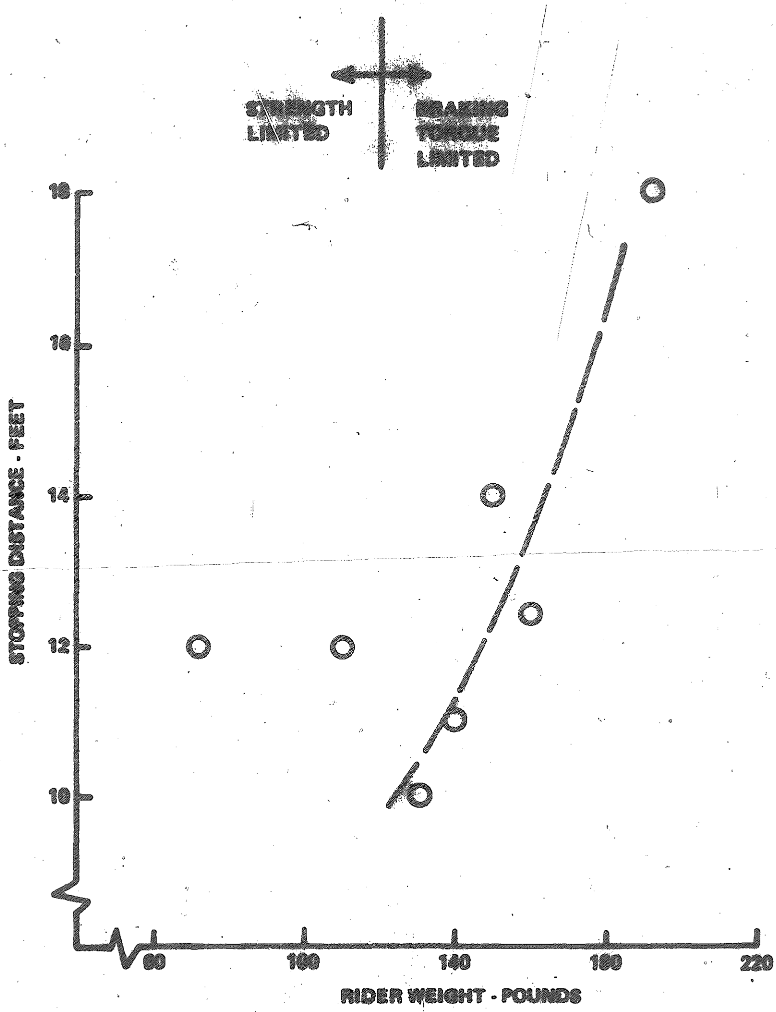
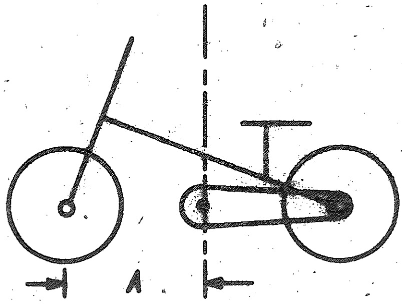


Figure 6 EFFECT OF RIDER WEIGHT ON STOPPING DISTANCE FOR BOTH BICYCLES



smaller than for the conventional unit (neglecting the effect of bicycle center-of-gravity location which also tends to favor the conventional bicycle). When the bicycles were operated in this mode, both could occasionally be made to produce complete pitchover by application of the front caliper brakes. The high rise bicycle was more sensitive in this respect than was the conventional design, pitchover occurring at speeds as low as 10 MPH on dry surfaces with high coefficients of friction. In over 200 hard braking stops made during the test program, however, no instance of pitchover (or even lifting of the rear wheel) occurred when the rider remained seated throughout the braking period.

Specific tests to determine the curb-climbing limit of the two bicycles were not performed because of safety considerations. The high riser was tested with a 130 pound rider over a 1 inch by 1 inch board and successfully crossed this obstacle without loss of stability. The rider was able to maintain relatively straight path control in this test at all speeds. For tests of this type to produce good quantitative results without undue risk to the rider, sophisticated instrumentation, well beyond that employed for this brief program, is required. It is recommended that computer simulation methods be used for in-depth studies of this phenomenon.

Lateral Stability and Control

The general mathematical theory of the stability and control of bicycles is given in the Appendix and will not be discussed in detail here in connection with the test program. Instead, we will merely point out some conditions which must be satisfied for stability and show how these may influence the handling qualities of the bicycle.

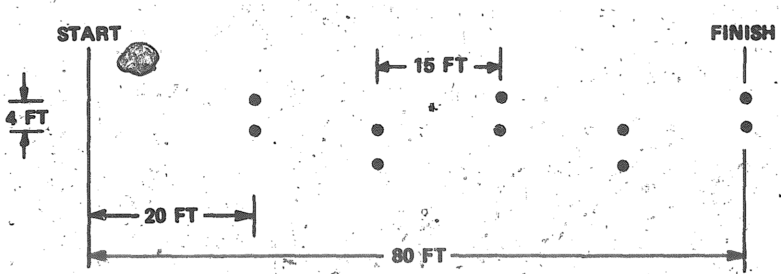
The bicycle, or any single track vehicle, must be stable in two interrelated degrees of motion freedom -- roll and yaw. (See Figure 3) Roll stability is most important at low speed; until the bicycle has reached sufficient speed for the gyroscopic moment of the spinning wheels to provide this stability, the rider must continually shift his weight from side to side to keep the bicycle upright. This is the primary problem faced by the beginning rider -- he is unwilling to operate at the higher speed needed to produce inherent stability yet lacks the experience to apply the necessary contributions to balance by weight-shifting. Hence, the bicycle usually falls over on its side and it is important that the rider be able to get free of the bicycle quickly and easily. As the rider gains experience and develops skill, he is able to forego some degree of stability in order to achieve higher maneuverability.

Three basic tests were performed to evaluate the stability and control of the two bicycles under study. The first of these was concerned with the inherent lateral stability which was examined by having the subjects ride the bicycles "hands-off" to determine the minimum speed at which a straight line path could be maintained. Results are shown in Table IV where it can be seen that in every case the high rise bicycle had to be pedaled at higher speed for path control to be maintained. Stable operation below the threshold speed requires a greater degree of rider compensation (i.e., weight-shifting).

TABLE IV: MINIMUM SPEED FOR "HANDS-OFF" STRAIGHT PATH FOLLOWING

Rider Weight	Bicycle	Minimum Hands-Off Speed MPH
70	Std	7
70	HR	10
140	Std	5
140	HR	7
190	HR	10
130	HR	10
130	Std	6
150	HR	12
150	Std	6

In a simple serpentine maneuvering test on a pattern as sketched below, no significant difference between the bicycles in the time required to complete the course was measured. This test is very definitely related to driver skill; it requires the bicycle to be banked over nearly to its limit for successful performance at an average speed of 14 MPH.



A plot of time required as a function of run number shown in Figure 7 demonstrates the learning process associated with this task. Note that the riders were able to perform this test equally well with each bicycle. In comparison tests in which the pylons were spaced at intervals of 7 1/2 feet, no significant difference between the bicycles was detected in a rider group of four subjects.

Maximum lateral acceleration capability was measured in steady state tests on a 50 ft. radius circle. The results obtained by a single skilled rider were within 10% of a mean value of .35g for both bicycles. A second rider reached a value of .335g with the conventional unit. Average speed for this test was approximately 16.5 MPH. It is concluded that both bicycles offer good cornering capability on dry surfaces in the normal speed range.

3.3 TEST RESULTS

Listed below are the principal results of the performance and handling tests:

1. Coaster brakes and rear caliper brakes are equally effective in panic stops. For riders in the 100 pound weight class, average deceleration values on dry asphalt are about .35g from an initial speed of about 15 MPH.

2. Under similar conditions, front caliper brakes are approximately 30% more efficient. That is, average deceleration values rise to about .45g. From 15 MPH, this represents a reduction in stopping distance of about 5 feet.

3. A slight additional improvement in stopping distance is achieved when both caliper brakes are applied. But this effect reduces stopping distance by, at most, two feet for the conditions cited above.

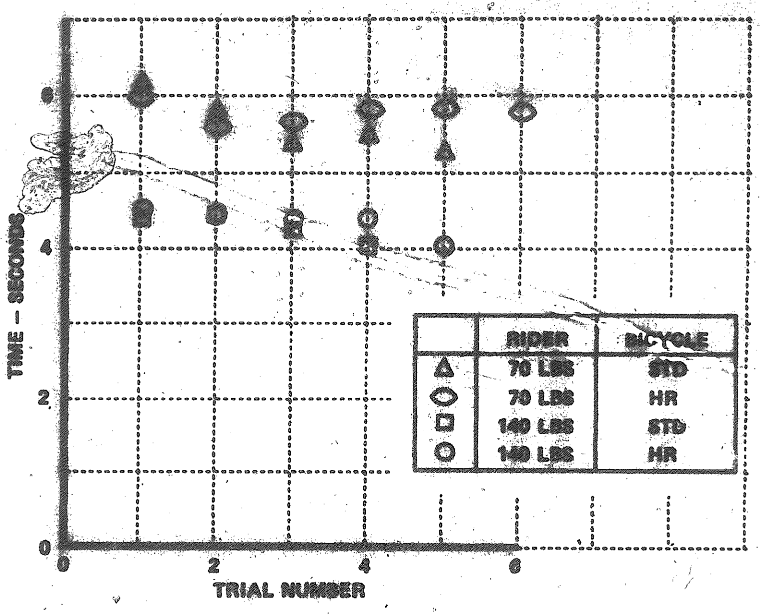


Figure 7 SERPENTINE TEST RESULTS

4. Under no conditions, as long as the rider remained seated, could a pitchover be induced in either bicycle.

5. Rear wheel braking is essentially self-stabilizing from the standpoint of front pitchover potential. However, when the rear brake is locked up, the bicycle becomes laterally unstable.

6. The low speed stability of the standard bicycle is superior to that of the high riser. This conclusion is based on a test to determine the minimum speed at which the bicycle is hands-off stable. Although absolute values of this speed varied with rider skill, the standard bicycle could be successfully ridden, hands-off, at speeds from 2 to 5 MPH slower than the high riser.

7. Lateral acceleration limit while pedaling on a high coefficient of friction surface is approximately the same for both bicycles - about 1/3g. In this mode of operation the limit is due to pedal clearance requirements which limits camber angle to about 20 degrees. At this condition, the tires of neither bicycle appear to be near their maximum cornering force.

If these results are interpreted in terms of safety of operation, the following conclusions can be drawn:

1. Front wheel brakes, regardless of bicycle size, can lead to problems in safe operation.

2. At the risk of incurring the wrath of millions of small boys, learners and novice riders should use girls-style bikes.

3. Bicycle designs making use of small front wheels must include compensating design features to avoid reducing the limits of safe operation.

4. Design features which can be impacted by the rider or which can compromise the rider's ability to get free of the bike in the event of an accident should be avoided.

3.4 MATHEMATICAL MODEL DEVELOPMENT

A digital computer simulation of a bicycle and rider has been developed. This simulation program is capable of predicting the transient handling qualities of bicycles of different designs for various maneuvers and different rider weights. The bicycle model is a system of three rigid masses with eight degrees of freedom as indicated in Figure 8. A rotating coordinate system is fixed in the rear wheel and frame structure and has three translational and three rotational degrees of freedom with respect to the space fixed coordinate system. The front wheel and steering fork has a rotational degree of freedom about an inclined steer axis. The rider has a rotational degree of freedom about a horizontal roll axis. The equations of motion include all the inertial coupling terms between the rider, the front wheel and steering fork and the rear wheel and frame. Included in the mathematical analysis are radial tire stiffness, tire side forces due to slip angle and camber angle, and the gyroscopic effects of the rotating wheels. A complete description of the mathematical model on which the simulation is based is given in Appendix 1.

Limitations of time and funding have prevented a thorough proof and validation study of the simulation. Perhaps the most essential of the missing information is tire performance data. The preliminary test work has enabled us to make first-order estimates of the tire factors which are needed for exercising the simulation but reliable values are needed before a meaningful parameter evaluation study can be undertaken. The required data include:

- Side force due to inclination angle as a function of normal force and tire pressure
- Side force due to slip angle as a function of normal force and tire pressure
- Aligning torque values.

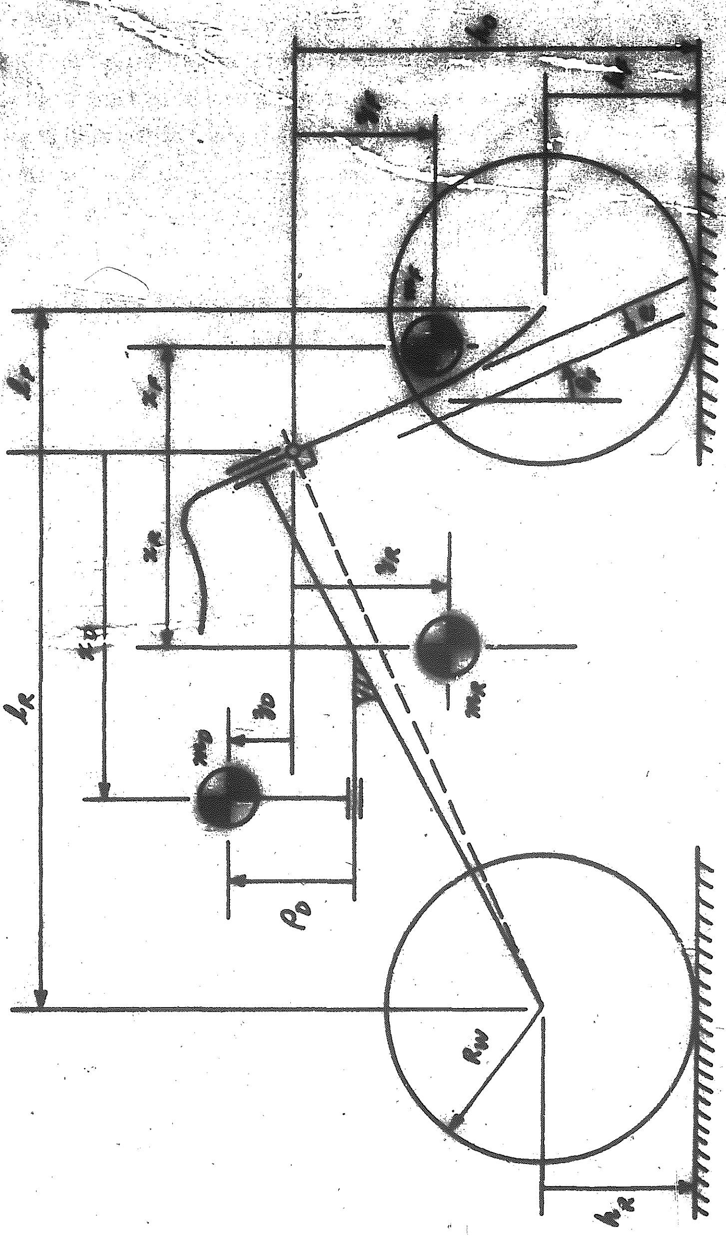


Figure 8 CHARACTERISTIC DIMENSIONS OF MODEL BICYCLE

As part of the current program, a preliminary design of a simple on-road bicycle tire tester has been generated. This unit, which could be attached to an unmodified full-size station wagon, would be capable of steady state measurements of the parameters listed above in the freely rolling mode. It is believed that this information would prove not only useful for the simulation studies but could be the key to improved bicycle design.

4.0 RECOMMENDATIONS

This preliminary study has served to identify certain design factors that are significant in bicycle performance and handling. To a lesser degree, performance and handling characteristics have been related to safety operation. However, it has not been possible to investigate the operational effect of a wide range of values for these parameters in this limited program. This would seem to be a desirable first step for the industry to take toward the development of performance standards and consumer information. The following recommendations are made:

1. Additional full-scale experimental work is recommended in order to build up an information bank on performance and handling characteristics.

2. Accident causation studies, linking design characteristics and safety, should be undertaken.

3. The mathematical model developed in this study should be used for further study of bicycle stability and control.

APPENDIX

DYNAMICAL ANALYSIS OF A BICYCLE

- Description of the Mathematical Model -

By: R. Douglas Roland, Jr.

INTRODUCTION

The mathematical analysis described herein is the basis of a digital computer simulation of a bicycle and rider. This computer program is capable of simulating the transient response of the bicycle to driver inputs. All the important design parameters of the bicycle are variable in the program, as well as rider weight and weight distribution, Figure 1. The program may be operated with no rider torque inputs to the front steering fork to simulate "hands off" riding or with rider control inputs in terms of torque on the steering fork and the roll moment between rider and frame.

The bicycle-rider model is a system of three rigid masses with eight degrees of freedom. The vehicle coordinate system (X, Y, Z) is defined as fixed in the rear wheel and frame structure and has three translational (u_o, v_o, w_o) and three rotational degrees (ψ, ϕ, θ) of freedom with respect to the space fixed coordinate system (X', Y', Z'), Figure 2. The front wheel and steering fork has a rotational degree of freedom (δ) about an inclined steer axis. The rider has a rotational degree of freedom (ϕ_p) about a horizontal roll axis, Figure 3. The origin of the vehicle coordinate system is at the intersection of the steering fork axis and the line which is perpendicular to the steering fork axis and passes through the rear wheel center. The X axis is along the longitudinal axis of the bicycle and is positive forward, the Y axis is positive to the right of the bicycle, and the Z axis is positive downward. The origin of the front fork coordinate system (X'', Y'', Z'') coincides with the origin of the vehicle coordinate system. The Z'' axis is coincident with the steering fork axis and positive downward, the X'' axis is positive forward, and the Y'' axis is positive to the right. The Y and Y'' axes are coincident at zero steer angle. The space fixed coordinate system is defined with the $X'-Y'$ plane as the ground plane and with the Z' axis parallel to the gravity vector and positive downward.

The mass distribution is assumed to be symmetrical with respect to the vertical-longitudinal plane through the geometrical center of the bicycle.

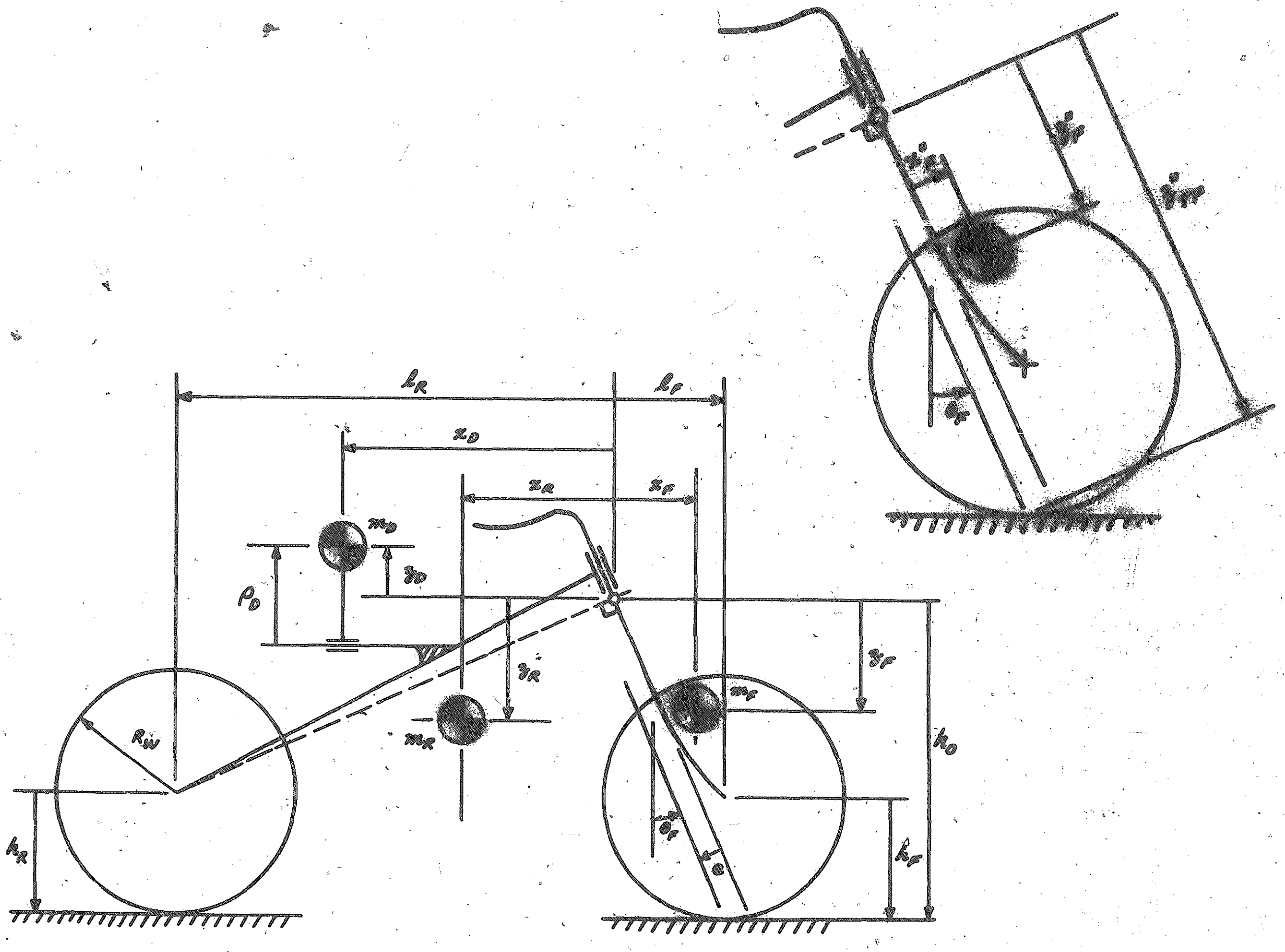


Figure 1 CHARACTERISTIC DIMENSIONS OF MODEL BICYCLE

Thus the $X-Y$ and $Y-Z$ products of inertia of each rigid mass are zero. All inertial coupling terms between the rider, the front wheel and steering fork, and the rear wheel and frame structure are included in the equations of motion. An important inclusion is the gyroscopic coupling effect of the rotating wheels.

It is assumed that the only external forces acting on the bicycle are the tire normal forces and tire side forces. The radial stiffness of the tires is assumed to be constant with rolling radius. The tire side forces are given as second order functions of normal load, slip angle, and camber angle. The tire slip angles are calculated from the components of the forward and lateral velocities of the ground contact point in the ground plane.

This mathematical analysis results in the formulation of the eight equations of motion for the bicycle and rider written in as a single matrix equation in Section 7. This matrix equation of motion includes all the inertial coupling between the rigid masses as well as the gyroscopic moment due to the rotating wheels. The only simplification to the equation being the small angle assumption for the front steer angle and the rider roll angle ($\sin\delta = \delta$, $\cos\delta = 1$, $\sin\phi_D = \phi_D$, $\cos\phi_D = 1$).

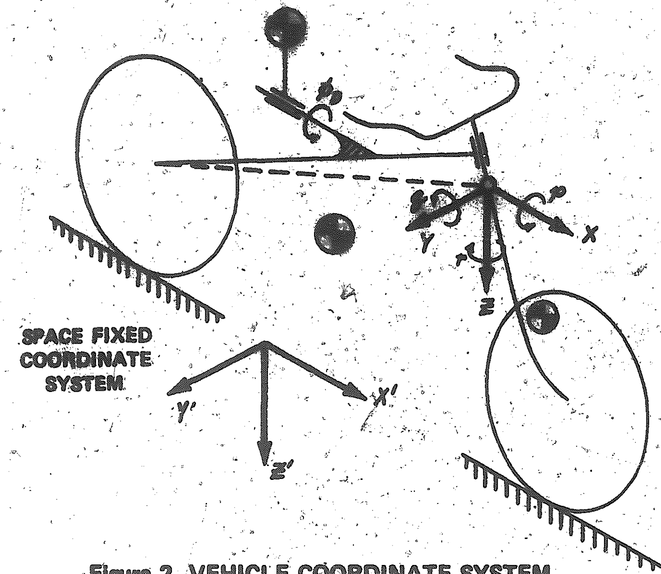


Figure 2 VEHICLE COORDINATE SYSTEM

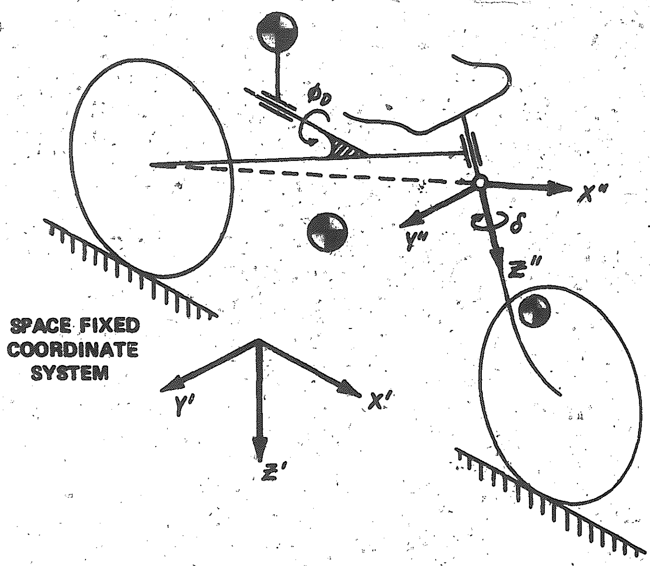


Figure 3 FRONT FORK COORDINATE SYSTEM

Section 1

REVIEW OF THE FUNDAMENTAL EQUATIONS FOR A ROTATING FRAME OF REFERENCE

The following review gives the general equations for the velocity and acceleration of a point and the general equation of rotational motion of a body in vector and matrix form for a rotating frame of reference.

The velocity of a point in a translating and rotating frame of reference is given by the following equation.

$$\bar{V} = \bar{V}_0 + \frac{\partial \bar{R}}{\partial t} + \bar{\Omega} \times \bar{R} \tag{1-1}$$

where:

$\bar{V}$ is the velocity of the point relative to the space fixed coordinate system and expressed in the coordinates of the rotating reference frame.

$\bar{V}_0$ is the velocity of the origin of the rotating frame of reference with respect to the space coordinate system and expressed in the coordinates of the rotating reference frame.

$\bar{R}$ is the position vector of the point relative to the origin of the rotating frame of reference and expressed in the coordinates of the rotating reference frame.

$\frac{\partial \bar{R}}{\partial t}$ is the velocity of the point relative to the rotating frame of reference and expressed in the coordinates of the rotating reference frame.

$\bar{\Omega}$ is the vector rotational velocity of the rotating frame of reference relative to the space fixed coordinate system and expressed in the coordinates of the rotating reference frame.

Equation 1-1 can be written in matrix form. This is the matrix equation for the velocity of a moving point in a translating and rotating frame of reference expressed in the coordinates of the rotating reference frame.

$$\begin{Vmatrix} u \\ v \\ w \end{Vmatrix} = \begin{Vmatrix} u_0 \\ v_0 \\ w_0 \end{Vmatrix} + \begin{Vmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Vmatrix} + \begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \times \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} \qquad (1\text{-}2)$$

Written in expanded form the above equation is:

$$\begin{Vmatrix} u \\ v \\ w \end{Vmatrix} = \begin{Vmatrix} u_0 \\ v_0 \\ w_0 \end{Vmatrix} + \begin{Vmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Vmatrix} + \begin{Vmatrix} qz - ry \\ -pz + rx \\ py - qx \end{Vmatrix} \qquad (1\text{-}3)$$

where:

x, y, z are the position coordinates of the point along the X, Y, Z axes of the rotating coordinate system.

$\dot{x}, \dot{y}, \dot{z}$ are the velocity components of the point relative to the rotating frame of reference along the X, Y, Z axes of the rotating coordinate system.

p, q, r are the rotational velocity components of the rotating frame of reference relative to the space fixed coordinate system around the X, Y, Z axes of the rotating coordinate system.

u_0, v_0, w_0 are the velocity components of the origin of the rotating frame of reference relative to the space fixed coordinate system along the X, Y, Z axes of the rotating coordinate system.

u, v, w are the velocity components of the point relative to the space fixed coordinate system along the X, Y, Z axes of the rotating coordinate system.

The acceleration of a point in a translating and rotating frame of reference is given by the following equation.

$$\dot{\bar{V}} = \frac{\partial \bar{V}}{\partial t} + \bar{\Omega} \times \bar{V} \tag{1-4}$$

where:

$\dot{\bar{V}}$ is the acceleration of the point relative to the space fixed coordinate system and expressed in the coordinates of the rotating reference frame.

Equation 1-1 may be substituted into Equation 1-4 yielding:

$$\dot{\bar{V}} = \frac{\partial}{\partial t}\left(\bar{V}_0 + \frac{\partial \bar{R}}{\partial t} + \bar{\Omega} \times \bar{R}\right) + \bar{\Omega} \times \left(\bar{V}_0 + \frac{\partial \bar{R}}{\partial t} + \bar{\Omega} \times \bar{R}\right) \tag{1-5}$$

or

$$\dot{\bar{V}} = \frac{\partial \bar{V}_0}{\partial t} + \bar{\Omega} \times \bar{V}_0 + \frac{\partial^2 \bar{R}}{\partial t^2} + \frac{\partial \bar{\Omega}}{\partial t} \times \bar{R} + \bar{\Omega} \times \left(\bar{\Omega} \times \bar{R}\right) + 2\bar{\Omega} \times \frac{\partial \bar{R}}{\partial t}$$

The above equation can be written in matrix form. This is the matrix equation for the inertial acceleration of a moving point in a translating and rotating frame of reference expressed in the coordinates of the rotating reference frame.

$$\begin{Vmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Vmatrix} = \begin{Vmatrix} \dot{u}_0 \\ \dot{v}_0 \\ \dot{w}_0 \end{Vmatrix} + \begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \times \begin{Vmatrix} u_0 \\ v_0 \\ w_0 \end{Vmatrix} + \begin{Vmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Vmatrix} + \begin{Vmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Vmatrix} \times \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} + \begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \times \left(\begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \times \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} \right) + 2 \begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \times \begin{Vmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Vmatrix} \tag{1-6}$$

Written in expanded form the above equation is as follows.

$$\begin{Vmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Vmatrix} = \begin{Vmatrix} \dot{u}_0 \\ \dot{v}_0 \\ \dot{w}_0 \end{Vmatrix} + \begin{Vmatrix} qw_0 - rv_0 \\ -pw_0 + ru_0 \\ pv_0 - qu_0 \end{Vmatrix} + \begin{Vmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Vmatrix} + \begin{Vmatrix} \dot{q}z - \dot{r}y \\ -\dot{p}z + \dot{r}x \\ \dot{p}y - \dot{q}x \end{Vmatrix}$$

$$+ \begin{Vmatrix} -(q^2 + r^2)x + pqy + prz \\ pqx - (p^2 + r^2)y + qrz \\ prx + qry - (p^2 + q^2)z \end{Vmatrix} + 2 \begin{Vmatrix} q\dot{z} - r\dot{y} \\ -p\dot{z} + r\dot{x} \\ p\dot{y} - q\dot{x} \end{Vmatrix} \qquad (1\text{-}7)$$

where:

$\ddot{x}, \ddot{y}, \ddot{z}$ are the acceleration components of the point relative to the rotating frame of reference along the X, Y, Z axes of the rotating coordinate system.

$\dot{p}, \dot{q}, \dot{r}$ are the rotational acceleration components of the rotating frame of reference relative to the space fixed coordinate system around the X, Y, Z axes of the rotating coordinate system.

$\dot{u}_0, \dot{v}_0, \dot{w}_0$ are the acceleration components of the origin of the rotating frame of reference relative to the space fixed coordinate system along the X, Y, Z axes of the rotating coordinate system.

$\dot{u}, \dot{v}, \dot{w}$ are the acceleration components of the point relative to the space fixed coordinate system along the X, Y, Z axes of the rotating coordinate system.

The equation for rotational motion of a body in a rotating coordinate system is written below in vector notation. This equation states that the rate of change of angular momentum is equal to the applied external moment.

$$\frac{d\bar{H}}{dt} + \bar{\Omega} \times \bar{H} = \bar{N} \tag{1-8}$$

where:

$\bar{H}$ is the vector angular momentum of the body expressed in the rotating coordinate system.

$\bar{\Omega}$ is the vector angular velocity of the rotating coordinate system.

$\bar{N}$ is the vector external moment acting on the body expressed in the rotating coordinate system.

The angular momentum vector may be expressed in terms of the total angular velocity of the body, $\bar{\omega}$, and its inertia tensor, $\bar{I}$.

$$\bar{H} = \bar{I}\,\bar{\omega} \tag{1-9}$$

Thus the vector equation for rotational motion may be written as follows.

$$I\frac{d\bar{\Omega}}{dt} + \frac{d\bar{I}}{dt}\bar{\Omega} + \bar{\Omega} \times (\bar{I}\,\bar{\omega}) = \bar{N} \tag{1-10}$$

The above equation is now written in matrix form excluding the term with the time derivative of the inertia tensor.

$$\begin{Vmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{Vmatrix} \begin{Vmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Vmatrix} + \begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \times \left(\begin{Vmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{Vmatrix} \begin{Vmatrix} p+\omega_x \\ q+\omega_y \\ r+\omega_z \end{Vmatrix} \right) = \begin{Vmatrix} N_x \\ N_y \\ N_z \end{Vmatrix} \qquad (1\text{-}11)$$

where:

I_x, I_y, I_z, I_{xy}, I_{xz}, I_{yz} are the moments and products of inertia of the body with respect to the axes of the rotating coordinate system.

$\omega_x, \omega_y, \omega_z$ are the rotational velocity components of the body relative to the rotating coordinate system along the X, Y, Z axes of the rotating coordinate system.

N_x, N_y, N_z are the components of the external moment on the body along the X, Y, Z axes of the rotating coordinate system.

Since the time derivative of the inertia tensor was assumed to be zero, Equation 1-11 is valid only if the moments and products of inertia of the body are fixed with respect to the rotating coordinate system. Therefore, if any of the angular velocity components ω_x, ω_y, ω_z are nonzero, the body must be symmetrical about the respective axes.

Equation 1-11 is now written in expanded form.

$$
\left\| \begin{matrix} \dot{p} I_X - \dot{q} I_{XY} - \dot{r} I_{XZ} \\ \\ -\dot{p} I_{XY} + \dot{q} I_Y - \dot{r} I_{YZ} \\ \\ -\dot{p} I_{XZ} - \dot{q} I_{YZ} + \dot{r} I_Z \end{matrix} \right\| + \left\| \begin{matrix} -q(p+\omega_X) I_{XZ} - \left[q(q+\omega_Y) - r(r+\omega_Z) \right] I_{YZ} \\ + q(r+\omega_Z) I_Z + r(p+\omega_X) I_{XY} - r(q+\omega_Y) I_Y \\ \\ \left[p(p+\omega_X) - r(r+\omega_Z) \right] I_{XZ} + p(q+\omega_Y) I_{YZ} \\ - p(r+\omega_Z) I_Z + r(p+\omega_X) I_X - r(q+\omega_Y) I_{XY} \\ \\ -\left[p(p+\omega_X) - q(q+\omega_Y) \right] I_{XY} + p(q+\omega_Y) I_Y \\ - p(r+\omega_Z) I_{YZ} - q(p+\omega_X) I_X + q(r+\omega_Z) I_{XZ} \end{matrix} \right\| = \left\| \begin{matrix} N_X \\ \\ N_Y \\ \\ N_Z \end{matrix} \right\| \quad (1\text{-}12)
$$

The position of the vehicle in the space coordinate system is defined by the coordinates of the sprung mass c.g. (x'_{G}, y'_{G}, z'_{G}) and the orientation of the vehicle in the space fixed coordinate system is defined by the Euler angles (ϕ, θ, ψ), about the X, Y, Z axes of the vehicle coordinate system taken in the order ψ, θ, ϕ.

The transformation matrix for transforming coordinates in the vehicle coordinate system into coordinates in the space fixed coordinate system is given below.

$$
\| A \| = \left\| \begin{array}{c|c|c} \cos\theta \cos\psi & -\cos\phi \sin\psi + \sin\phi \sin\theta \cos\psi & \sin\phi \sin\psi + \cos\phi \sin\theta \cos\psi \\ \hline \cos\theta \sin\psi & \cos\phi \cos\psi + \sin\phi \sin\theta \sin\psi & -\cos\psi \sin\phi + \cos\phi \sin\theta \sin\psi \\ \hline -\sin\theta & \cos\theta \sin\phi & \cos\theta \cos\phi \end{array} \right\| \quad (1\text{-}13)
$$

Section 2
WHEEL INCLINATION AND SLIP ANGLES

The inclination angles and slip angles of the front and rear wheels are determined so that tire forces and moments at the ground contact points may be obtained.

The direction cosines in the space fixed coordinate system of the normal to the front wheel plane are found below. The elements of the right most matrix represent the direction cosines of the normal to the wheel plane in vehicle coordinate system.

$$\begin{Vmatrix} \cos\alpha_{YWF} \\ \cos\beta_{YWF} \\ \cos\gamma_{YWF} \end{Vmatrix} = \begin{Vmatrix} A \end{Vmatrix} \cdot \begin{Vmatrix} -\sin\delta\cos\theta_F \\ \cos\delta \\ \sin\delta\sin\theta_F \end{Vmatrix} \tag{2-1}$$

where:

$\cos\alpha_{YWF}$, $\cos\beta_{YWF}$, $\cos\gamma_{YWF}$ are the direction cosines of the normal to the front wheel plane with respect to the X', Y', Z' axes of the space fixed coordinate system.

δ is the rotation of the front wheel about the steer axis.

θ_F is the caster angle of the front wheel steer axis.

The inclination angle of the front wheel with respect to the ground, ϕ_F, is the angle between the wheel plane and a plane perpendicular to the ground plane and having the same line of intersection with the ground plane as the wheel plane.

$$\phi_F = \arcsin\left(\cos\alpha_{YWF}\cos\alpha_G + \cos\beta_{YWF}\cos\beta_G + \cos\gamma_{YWF}\cos\gamma_G\right) \tag{2-2}$$

where

$\cos \alpha_G$, $\cos \beta_G$, $\cos \gamma_G$ are the direction cosines of the normal to the ground plane with respect to the X', Y', Z' axes of the space fixed coordinate system.

Since flat level ground is assumed the normal to the ground plane is vertical and $\cos \alpha_G = 0$, $\cos \beta_G = 0$, $\cos \gamma_G = 1$.

therefore $\phi_F = \arcsin(\cos \gamma_{YWF})$ (2-3)

The direction cosines in the space fixed coordinate system of the normal to the rear wheel plane are found below.

$$\begin{Vmatrix} \cos \alpha_{YWR} \\ \cos \beta_{YWR} \\ \cos \gamma_{YWR} \end{Vmatrix} = \|A\| \cdot \begin{Vmatrix} 0 \\ 1 \\ 0 \end{Vmatrix} \tag{2-4}$$

Again assuming flat level ground the inclination angle of the rear wheel with respect to the ground, ϕ_R, may be expressed as follows.

$$\phi_R = \arcsin(\cos \gamma_{YWR}) \tag{2-5}$$

The slip angle is the angle in the ground plane between the heading vector of the wheel and the velocity vector of the ground contact point. Since the ground plane is parallel to the X'-Y' plane of the space fixed coordinate system, the heading angle is the angle between the X' axis and the line of intersection of wheel plane and the ground plane.

Thus the front and rear wheel heading angles, ψ_F and ψ_R, are as follows.

$$\psi_F = \arctan(-\cos \alpha_{YWF} / \cos \beta_{YWF}) \tag{2-6}$$

$$\psi_R = \arctan(-\cos \alpha_{YWR} / \cos \beta_{YWR}) \tag{2-7}$$

It is assumed that changes in the rolling radii of the front and rear wheels and that the lateral velocity of the front wheel contact point with respect to the vehicle are negligible. Equation 1-3 is now solved for the velocity components (u_F, v_F, w_F and u_R, v_R, w_R) along the X, Y, Z axes of points located at the front and rear wheel ground contact points but fixed in the vehicle coordinate system.

$$u_F = u_0 + q h_0 \tag{2-8}$$

$$u_R = u_0 + q h_0 \tag{2-9}$$

$$v_F = v_0 - p h_0 + r l_F \tag{2-10}$$

$$v_R = v_0 - p h_0 + r l_R \tag{2-11}$$

$$w_F = w_0 - q l_F \tag{2-12}$$

$$w_R = w_0 - q l_R \tag{2-13}$$

The components in the space fixed coordinate system of the velocity vectors of the front and rear wheel ground contact points are found below.

$$\begin{Vmatrix} u'_F \\ v'_F \\ w'_F \end{Vmatrix} = \begin{Vmatrix} A \end{Vmatrix} \cdot \begin{Vmatrix} u_F \\ v_F \\ w_F \end{Vmatrix} \tag{2-14}$$

$$\begin{Vmatrix} u'_R \\ v'_R \\ w'_R \end{Vmatrix} = \begin{Vmatrix} A \end{Vmatrix} \cdot \begin{Vmatrix} u_R \\ v_R \\ w_R \end{Vmatrix} \tag{2-15}$$

Since the ground plane is parallel to the X'-Y' plane of the space fixed coordinate system, the angle between the X' axis and the velocity vectors of the front and rear wheel ground contact points in the ground plane may be found as follows.

$$\psi_{VF} = \arctan \left(v'_F / u'_F \right) \tag{2-16}$$

$$\psi_{VR} = \arctan \left(v'_R / u'_R \right) \tag{2-17}$$

The front and rear wheel slip angles may now be expressed as follows.

$$\alpha_F = \psi_F - \psi_{VF} \tag{2-18}$$

$$\alpha_R = \psi_R - \psi_{VR} \tag{2-19}$$

Section 3
TIRE FORCES

The instantaneous rolling radii are defined as the shortest distance in the wheel plane from the wheel center to the ground plane.

$$h_F = -\sec \phi_F \, z'_{WF} \tag{3-1}$$

Where z'_{WF} is the elevation of the front wheel center above the ground plane and may be found from the following equation.

$$\begin{Vmatrix} x'_{WF} \\ y'_{WF} \\ z'_{WF} \end{Vmatrix} = \begin{Vmatrix} x'_0 \\ y'_0 \\ z'_0 \end{Vmatrix} + \begin{Vmatrix} A \end{Vmatrix} \begin{Vmatrix} \ell_F \\ 0 \\ h_0 - R_W \end{Vmatrix} \tag{3-2}$$

The instantaneous rolling radius of the rear wheel is found below.

$$h_R = -\sec \phi_R \, z'_{WR} \tag{3-3}$$

where

$$\begin{Vmatrix} x'_{WR} \\ y'_{WR} \\ z'_{WR} \end{Vmatrix} = \begin{Vmatrix} x'_0 \\ y'_0 \\ z'_0 \end{Vmatrix} + \begin{Vmatrix} A \end{Vmatrix} \begin{Vmatrix} \ell_R \\ 0 \\ h_0 - R_W \end{Vmatrix} \tag{3-4}$$

The radial force on a wheel acts along the line connecting the ground contact point and the wheel center. The radial force on the front or rear wheel, F_{RF} or F_{RR}, is equal to the radial stiffness of the tire times the radial deflection.

$$F_{RF} = K_R(R_W - h_F) \tag{3-5}$$
$$F_{RR} = K_R(R_W - h_R) \tag{3-6}$$

where:

K_R is the radial stiffness of a tire (assumed to be the same for front and rear tires).

R_W is the undeflected rolling radius of the wheel.

h_F, h_R are the instantaneous rolling radii of the front and rear wheels.

If a wheel is not in contact with the ground the radial force is zero thus the tire forces and moments are zero for that wheel. If the radial force is not zero, the normal forces can be found from the following equations (Figure 4).

$$F_{NF} = F_{RF} \sec \phi_F + F_{SF} \tan \phi_F \tag{3-7}$$

$$F_{NR} = F_{RR} \sec \phi_R + F_{SR} \tan \phi_R \tag{3-8}$$

where F_{SF}, F_{SR} are the side forces on the front and rear tires. The side forces on the front and rear tires are expressed by the following approximate relationships.

$$F_{SF} = F_{NF} \left(K_{\alpha 1} \alpha_F + K_{\alpha 2} \alpha_F^2 + K_{\phi 1} \phi_F + K_{\phi 2} \phi_F^2\right) \tag{3-9}$$

$$F_{SR} = F_{NR} \left(K_{\alpha 1} \alpha_R + K_{\alpha 2} \alpha_R^2 + K_{\phi 1} \phi_R + K_{\phi 2} \phi_R^2\right) \tag{3-10}$$

where: $K_{\alpha 1}, K_{\alpha 2}$ are tire characteristics relating side force and slip angle.

$K_{\phi 1}, K_{\phi 2}$ are tire characteristics relating side force and inclination angle.

The overturning moment, rolling resistance moment, and aligning torque are assumed to be zero. The components of the front tire forces are now written in the front fork coordinate system.

$$F''_{XTF} = -\sin \theta_F \left(-\sin \phi_F F_{SF} + \cos \phi_F F_{NF}\right) \tag{3-11}$$

$$F''_{YTF} = \cos \phi_F F_{SF} + \sin \phi_F F_{NF} \tag{3-12}$$

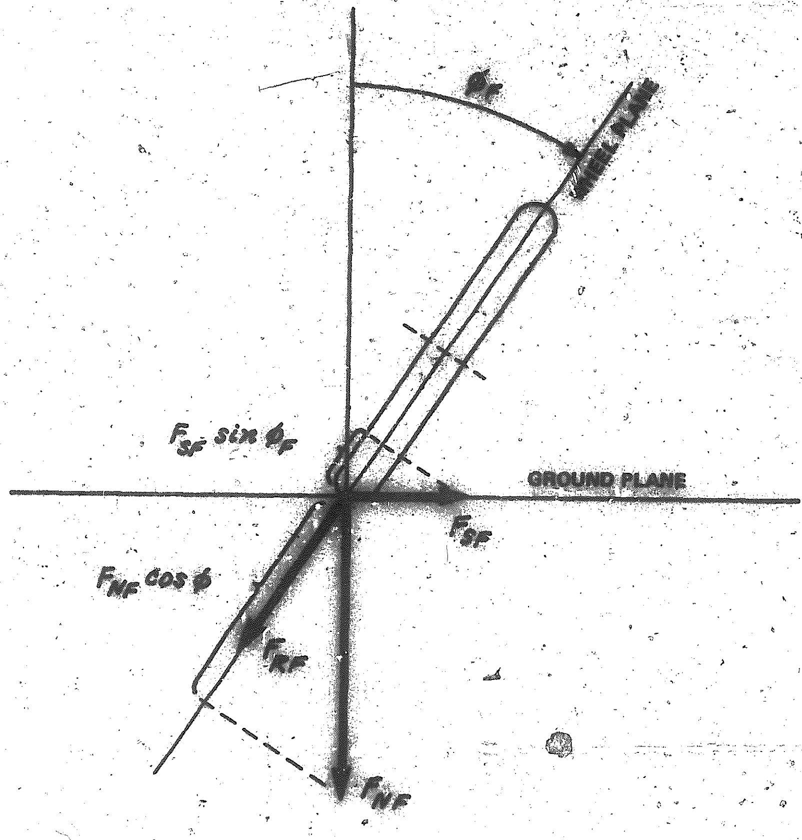
$$F''_{ZTF} = \cos \theta_F \left(-\sin \phi_F F_{SF} + \cos \phi_F F_{NF}\right) \tag{3-13}$$

Eliminating F_{NF} by using equation 3-7 the above equations reduce to the following.

$$F''_{XTF} = -\sin \theta_F F_{RF} \tag{3-14}$$

$$F''_{YTF} = \tan \phi_F F_{RF} + \sec \phi_F F_{SF} \tag{3-15}$$

$$F''_{ZTF} = \cos \theta_F F_{RF} \tag{3-16}$$



$$F_{RF} = F_{NF} \cos \phi_F - F_{SF} \sin \phi_F$$

$$\therefore F_{NF} = F_{RF} \sec \phi_F + F_{SF} \tan \phi_F$$

Figure 4 CALCULATION OF TIRE NORMAL FORCE

The components of the front tire forces in the vehicle coordinate system are given by the following equation.

$$\begin{Vmatrix} F_{XTF} \\ F_{YTF} \\ F_{ZTF} \end{Vmatrix} = \begin{Vmatrix} S \end{Vmatrix} \cdot \begin{Vmatrix} F''_{XTF} \\ F''_{YTF} \\ F''_{ZTF} \end{Vmatrix} \tag{3-17}$$

where $\| S \|$ is the transformation matrix from front fork coordinate system to the vehicle coordinate system.

The transformation matrix for transforming coordinates in the front fork coordinate system into coordinates in the vehicle coordinate system is given below.

$$\| S \| = \left\| \begin{array}{c:c:c} \cos\theta_F \cos\delta & -\cos\theta_F \sin\delta & \sin\theta_F \\ \hdashline \sin\delta & \cos\delta & 0 \\ \hdashline -\sin\theta_F \cos\delta & \sin\theta_F \sin\delta & \cos\theta_F \end{array} \right\| \tag{3-19}$$

and

$$\| S^T \| = \left\| \begin{array}{c:c:c} \cos\theta_F \cos\delta & \sin\delta & -\sin\theta_F \cos\delta \\ \hdashline -\cos\theta_F \sin\delta & \cos\delta & \sin\theta_F \sin\delta \\ \hdashline \sin\theta_F & 0 & \cos\theta_F \end{array} \right\| \tag{3-20}$$

The components of the rear tire forces are now written in the vehicle coordinate system.

$$F_{XTR} = 0 \tag{3-21}$$

$$F_{YTR} = \cos\phi_R F_{SR} + \sin\phi_R F_{NR} \tag{3-22}$$

$$F_{ZTR} = -\sin\phi_R F_{SR} + \cos\phi_R F_{NR} \tag{3-23}$$

Eliminating F_{NK} by using equation 3-8 the above equations reduce to the following.

$$F_{XTK} = 0 \tag{3-24}$$

$$F_{YTK} = \tan \phi_K F_{RK} + \sec \phi_K F_{SK} \tag{3-25}$$

$$F_{ZTK} = F_{ZK} \tag{3-26}$$

Section 4
FORCE AND MOMENT EQUATIONS FOR THE DRIVER

Equation 1-7 is now for the inertial acceleration components in the vehicle coordinate system of the driver c.g.

$$\dot{u}_D = \dot{u}_0 + q w_0 - r v_0 + \dot{q} z_D + \dot{r}\left(\rho_D \phi_D\right) - pq\left(\rho_D \phi_D\right) - \left(q^2 + r^2\right) x_D + pr z_D - 2q\dot{\phi}_D\left(\rho_D \phi_D\right) + 2r\dot{\phi}_D \rho_D \qquad (4\text{-}1)$$

$$\dot{v}_D = \dot{v}_0 - p w_0 + r u_0 - \dot{p} z_D + \dot{r} x_D + pq x_D + \left(p^2 + r^2\right)\left(\rho_D \phi_D\right) + qr z_D + 2p\dot{\phi}_D\left(\rho_D \phi_D\right) + \dot{\phi}_D^2\left(\rho_D \phi_D\right) - \ddot{\phi}_D \rho_D \qquad (4\text{-}2)$$

$$\dot{w}_D = \dot{w}_0 + p v_0 - q u_0 - \dot{p}\left(\rho_D \phi_D\right) - \dot{q} x_D + pr x_D - \left(p^2 + q^2\right) z_D - qr\left(\rho_D \phi_D\right) - 2p\dot{\phi}_D \rho_D - \dot{\phi}_D^2 \rho_D - \ddot{\phi}\left(\rho_D \phi_D\right) \qquad (4\text{-}3)$$

The force balance equations for the driver in the X, Y, Z directions are given below.

$$m_D \dot{u}_D = F_{XD} - m_D g \sin\theta \qquad (4\text{-}4)$$

$$m_D \dot{v}_D = F_{YD} + m_D g \cos\theta \sin\phi \qquad (4\text{-}5)$$

$$m_D \dot{w}_D = F_{ZD} + m_D g \cos\theta \cos\phi \qquad (4\text{-}6)$$

where

F_{XD}, F_{YD}, F_{ZD} are the X, Y, Z components of the reaction force acting on the driver at the driver roll center.

Thus the reaction forces acting on the driver may be written as follows.

$$F_{XD} = m_D \left(\dot{u}_0 + \dot{q} z_D + \dot{r} \left\{\rho_D \phi_D\right\}\right) - \gamma_{13} \tag{4-7}$$

$$F_{YD} = m_D \left(\dot{v}_0 - \dot{p} z_D + \dot{r} x_D - \ddot{\phi}_D \rho_D\right) - \gamma_{23} \tag{4-8}$$

$$F_{ZD} = m_D \left(\dot{w}_0 - \dot{p} \left\{\rho_D \phi_D\right\} - \dot{q} x_D - \ddot{\phi}_D \left\{\rho_D \phi_D\right\}\right) - \gamma_{33} \tag{4-9}$$

Where γ_{13}, γ_{23}, and γ_{33} are given in Equations 7-8, 7-9, and 7-10.

The inertia tensor for the driver about his c.g. is given below.

$$\left\| I_D \right\| = \left\| \begin{matrix} I_{DX} & 0 & -I_{DXZ} \\ 0 & I_{DY} & 0 \\ -I_{DXZ} & 0 & I_{DZ} \end{matrix} \right\| \tag{4-10}$$

I_{DX}, I_{DY} and I_{DZ} are respectively the X, Y, Z moments of inertia of the driver about his c.g. I_{DXZ} is the X-Z product of inertia about his c.g. Since the driver is assumed to be symmetrical about the X-Z plane, the X-Y and Y-Z products of inertia are zero.

The moment acting about the c.g. of the driver with respect to the coordinates fixed in the driver is given below in matrix form with the small angle assumptions for ϕ_D $(\sin\phi_D = \phi_D,\ \cos\phi_D = 1)$

$$\left\| N_D \right\| = \left\| \begin{matrix} N_{XD} + \rho_D (F_{YD} + \phi_D F_{ZD}) \\ N_{YD} + \phi_D N_{ZD} - \rho_D F_{XD} \\ -\phi_D N_{YD} + N_{ZD} \end{matrix} \right\| \tag{4-11}$$

Equation 1-11 is now written in matrix form for the driver. This is the moment balance equation for the driver about the axes of the coordinate system fixed in the driver.

$$\left\| I_D \right\| \left\| \begin{matrix} \dot{p} + \ddot{\phi}_D \\ \dot{q} + \phi_D \dot{r} \\ -\phi_D \dot{q} + \dot{r} \end{matrix} \right\| + \left\| \begin{matrix} p + \dot{\phi}_D \\ q + \phi_D r \\ -\phi_D q + r \end{matrix} \right\| \times \left(\left\| I_D \right\| \left\| \begin{matrix} p + \dot{\phi}_D \\ q + \phi_D r \\ -\phi_D q + r \end{matrix} \right\| \right) = \left\| N_D \right\| \tag{4-12}$$

Equation 4-12 may be expanded to the following.

$$\left\| \begin{matrix} (\dot{p} + \ddot{\phi}_D) I_{DX} - (-\phi_D \dot{q} + \dot{r}) I_{DXZ} - (q + \phi_D r)(p + \dot{\phi}_D) I_{DXZ} + \left(qr - \phi_D \left\{ q^2 - r^2 \right\}\right)(I_{DZ} - I_{DY}) \\ (\dot{q} + \phi_D \dot{r}) I_{DY} + (p + \dot{\phi}_D)(-\phi_D q + r)(I_{DX} - I_{DZ}) + \left(\left[p + \dot{\phi}_D \right]^2 + 2\phi_D q r - r^2 \right) I_{DXZ} \\ -(\dot{p} + \ddot{\phi}_D) I_{DXZ} + (-\phi_D \dot{q} + \dot{r}) I_{DZ} - (p + \dot{\phi}_D)(q + \phi_D r)(I_{DX} - I_{DY}) + \left(qr - \phi_D \left\{ q^2 - r^2 \right\}\right) I_{DXZ} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} N_{XD} + \rho_D (F_{YD} + \phi_D F_{ZD}) \\ N_{YD} + \phi_D N_{ZD} - \rho_D F_{XD} \\ -\phi_D N_{YD} + N_{ZD} \end{matrix} \right\| \tag{4-13}$$

From Equation 4-13, the equation of rotational motion of the driver about its roll axis is now written.

$$
\begin{aligned}
&-\rho_D m_D \dot{v}_0 - \phi_D \rho_D m_D \dot{w}_0 + (I_{DX} + \rho_D y_D m_D)\dot{p} + \phi_D (I_{DXZ} + \rho_D z_D m_D)\dot{q} \\
&-(I_{DXZ} + \rho_D z_D m_D)\dot{r} + (I_{DX} + \rho_D^2 m_D)\ddot{\phi}_D = (q + \phi_D r)(p + \dot{\phi}_D) I_{DXZ} \\
&+\left(qr - \phi_D \left\{q^2 - r^2\right\}\right)(I_{DY} - I_{DZ}) - \rho_D \gamma_{23} - \phi_D \rho_D \gamma_{33} + N_{XD}
\end{aligned} \qquad (4\text{-}14)
$$

The Y and Z components of the reaction moment on the driver at the driver roll center are written below.

$$
\begin{aligned}
N_{YD} = \phi_D \Big[&(\dot{p} + \ddot{\phi}_D) I_{DXZ} - (-\phi_D \dot{q} + \dot{r}) I_{DX} + (p + \dot{\phi}_D)(q + \phi_D r)(I_{DX} - I_{DY}) \\
&- \left(qr - \phi_D \left\{q^2 - r^2\right\}\right) I_{DXZ} \Big] + (\dot{q} + \phi_D \dot{r}) I_{DY} + (p + \dot{\phi}_D)(-\phi_D q + r)(I_{DX} - I_{DZ}) \\
&+ \left(\left\{p + \dot{\phi}_D\right\}^2 + 2\phi_D qr - r^2\right) I_{DXZ} + \rho_D m_D \left(\dot{u}_0 + \dot{q} z_D + \dot{r}\left\{\rho_D \phi_D\right\} - \rho_D \gamma_{13}\right)
\end{aligned} \qquad (4\text{-}15)
$$

$$
\begin{aligned}
N_{ZD} = \phi_D \Big[&(\dot{q} + \phi_D \dot{r}) I_{DY} + (p + \dot{\phi}_D)(-\phi_D q + r)(I_{DX} - I_{DZ}) + \left(\left\{p + \dot{\phi}_D\right\}^2 + 2\phi_D qr \right. \\
&\left. - r^2\right) I_{DXZ} \Big] + \phi_D \rho_D m_D \left(\dot{u}_0 + \dot{q} z_D + \dot{r}\left\{\rho_D \phi_D\right\}\right) - \phi_D \rho_D \gamma_{13} \\
&- (\dot{p} + \ddot{\phi}_D) I_{DXZ} + (-\phi_D \dot{q} + \dot{r}) I_{DZ} - (p + \dot{\phi}_D)(q + \phi_D r)(I_{DX} - I_{DY}) \\
&+ \left(qr - \phi_D \left\{q^2 - r^2\right\}\right) I_{DXZ}
\end{aligned} \qquad (4\text{-}16)
$$

Where γ_{13}, γ_{23}, and γ_{33} are given in Equations 7-11, 7-12 and 7-13.

Section 5

FORCE AND MOMENT EQUATIONS FOR FRONT WHEEL AND STEERING FORK

Equation 1-7 is now written for the inertial acceleration of the c.g. of the front wheel and steering fork in the vehicle coordinate system.

$$
\begin{Vmatrix} \dot u_F \\ \dot v_F \\ \dot w_F \end{Vmatrix} = \begin{Vmatrix} \dot u_0 + q w_0 - r v_0 \\ \dot v_0 - p w_0 + r u_0 \\ \dot w_0 + p v_0 - q u_0 \end{Vmatrix} + \begin{Vmatrix} \dot p + \ddot\delta \sin\theta_F \\ \dot q \\ \dot r + \ddot\delta\cos\theta_F \end{Vmatrix} \times \begin{Vmatrix} x_F \\ x_F''\delta \\ z_F \end{Vmatrix}
$$

$$
+ \begin{Vmatrix} p + \dot\delta\sin\theta_F \\ q \\ r + \dot\delta\cos\theta_F \end{Vmatrix} \times \begin{Vmatrix} p + \dot\delta\sin\theta_F \\ q \\ r + \dot\delta\cos\theta_F \end{Vmatrix} \times \begin{Vmatrix} x_F \\ x_F''\delta \\ z_F \end{Vmatrix} \tag{5-1}
$$

With the small angle assumption made for δ $(\sin\delta = \delta,\ \cos\delta = 1)$, the components of the acceleration are now written in expanded form.

$$
\begin{aligned}
\dot u_F = {} & \dot u_0 + q w_0 - r v_0 + z_F \dot q - x_F''\delta(\dot r + \ddot\delta\cos\theta_F) - x_F(q^2 + r^2) + x_F''\delta pq \\
& + z_F pr + z_F\cos\theta_F\, p\dot\delta - (x_F'' + x_F\cos\theta_F)\, r\dot\delta - x_F''\cos\theta_F\,\dot\delta^2 \\
& + x_F''\delta\sin\theta_F\, q\dot\delta
\end{aligned} \tag{5-2}
$$

$$
\begin{aligned}
\dot v_F = {} & \dot v_0 - p w_0 + r u_0 - z_F(\dot p + \ddot\delta\sin\theta_F) \pm x_F(\dot r + \ddot\delta\cos\theta_F) + x_F pq \\
& - x_F''\delta(p^2 + r^2) + z_F qr + z_F'' q\dot\delta - 2x_F''\delta\cos\theta_F\, r\dot\delta \\
& - 2x_F''\delta\sin\theta_F\, p\dot\delta - x_F''\delta\dot\delta^2
\end{aligned} \tag{5-3}
$$

$$
\begin{aligned}
\dot w_F = {} & \dot w_0 + p v_0 - q u_0 + x_F''\delta(\dot p + \ddot\delta\sin\theta_F) - x_F\dot q + x_F pr + x_F''\delta qr \\
& - z_F(p^2 + q^2) + (x_F'' - z_F\sin\theta_F)\, p\dot\delta + x_F''\delta\cos\theta_F\, q\dot\delta \\
& + x_F\sin\theta_F\, r\dot\delta + x_F''\sin\theta_F\,\dot\delta^2
\end{aligned} \tag{5-4}
$$

where

$$x_F'' = x_F \cos\theta_F - z_F \sin\theta_F \qquad (5\text{-}5)$$

$$z_F'' = x_F \sin\theta_F + z_F \cos\theta_F \qquad (5\text{-}6)$$

The force balance equations for the front wheel and steering fork in the X, Y, Z directions are given below.

$$m_F \dot{u}_F = F_{XTF} + F_{XF} - m_F g \sin\theta \qquad (5\text{-}7)$$

$$m_F \dot{v}_F = F_{YTF} + F_{YF} + m_F g \cos\theta \sin\phi \qquad (5\text{-}8)$$

$$m_F \dot{w}_F = F_{ZTF} + F_{ZF} + m_F g \cos\theta \cos\phi \qquad (5\text{-}9)$$

where

$F_{XTF}, F_{YTF}, F_{ZTF}$ are the X, Y, Z components of the tire force acting on the front wheel at the ground contact point.

F_{XF}, F_{YF}, F_{ZF} are the X, Y, Z components of the reaction force acting on the steering fork at the origin of the vehicle coordinate system.

Thus the reaction forces acting on the front wheel and steering fork may be written as follows.

$$F_{XF} = m_F (\dot{u}_0 + z_F \dot{q} - x_F'' \delta \dot{r} - x_F'' \delta \cos\theta_F \ddot{\delta}) - \dot{d}_{12} - F_{XTF} \qquad (5\text{-}10)$$

$$F_{YF} = m_F (\dot{v}_0 - z_F \dot{p} + x_F \dot{r} + x_F'' \ddot{\delta}) - \dot{d}_{22} - F_{YTF} \qquad (5\text{-}11)$$

$$F_{ZF} = m_F\left(\dot{w}_0 + x''_F\,\delta\,\dot{p} - x_F\,\dot{q} + x''_F\,\delta\,\sin\varphi_0\,\ddot{\delta}\right) - \gamma_{32} - F_{ZTF} \qquad (5\text{-}12)$$

Where γ_{12}, γ_{22}, γ_{32} are given in Equations 7-11, 7-12 and 7-13.

The moment on the front wheel which reacts the gyroscopic effect of the spinning wheel is given below in matrix form. It is assumed that the moment of inertia about a diameter through the wheel is equal to one half the moment of inertia about the spin axis, I_W.

$$\begin{Vmatrix} N''_{XWF} \\ N''_{YWF} \\ N''_{ZWF} \end{Vmatrix} = \begin{Vmatrix} p'' \\ q'' \\ r'' + \dot{\delta} \end{Vmatrix} \times \left(\begin{Vmatrix} \frac{I_{WF}}{2} & 0 & 0 \\ 0 & I_{WF} & 0 \\ 0 & 0 & \frac{I_{WF}}{2} \end{Vmatrix} \begin{Vmatrix} p'' \\ q'' + \omega_F \\ r'' + \dot{\delta} \end{Vmatrix} \right) \qquad (5\text{-}13)$$

where

N''_{XWF}, N''_{YWF}, N''_{ZWF} are the X", Y", Z" components of the moment on the front wheel which reacts the gyroscopic effect.

p'', q'', r'' are the X", Y", Z" components of angular velocity vector of the vehicle coordinate system.

$\dot{\delta}$ is the angular velocity of the front wheel and steering fork about the Z" axis with respect to the vehicle coordinate system.

ω_F is the spin velocity of the front wheel (assumed to be the forward velocity of the bicycle divided by the rolling radius of the wheel).

I_{WF} is the moment of inertia of the front wheel about its spin axis.

Assuming that the velocity q'' is negligible compared to the wheel spin velocity, ω_F, Equation 5-13 reduces to the following.

$$N''_{XWF} = -(r'' + \dot{\delta})\,\omega_F I_{WF} \tag{5-14}$$

$$N''_{YWF} = 0 \tag{5-15}$$

$$N''_{ZWF} = p''\,\omega_F I_{WF} \tag{5-16}$$

The inertia tensor for the front wheel and steering fork about its c.g. is given below.

$$\left\| I''_F \right\| = \left\| \begin{matrix} I''_{FX} & 0 & -I''_{FXZ} \\ 0 & I''_{FY} & 0 \\ -I''_{FXZ} & 0 & I''_{FZ} \end{matrix} \right\| \tag{5-17}$$

I''_{FX}, I''_{FY} and I''_{FZ} are respectively the X'', Y'', Z'' moments of inertia of the front wheel and steering fork about its c.g. I''_{FXZ} is the X''-Z'' product of inertia of the front wheel and steering fork about its c.g. Since the vehicle is assumed to be symmetrical about the X-Z plane (for $\delta = 0$), its X''-Y'' and Y''-Z'' products of inertia are zero.

The moment acting about the c.g. of the front wheel and steering fork with respect to the coordinate system fixed in the steering fork is given below in matrix form.

$$\left\| N''_F \right\| = \left\| \begin{matrix} N''_{XF} - N''_{XWF} + F''_{YF} z''_F - F''_{YTF}\left(z''_{TF} - z''_F\right) \\ N''_{YF} - N''_{YWF} - F''_{XF} z''_F + F''_{ZF} x''_F + F''_{XTF}\left(z''_{TF} - z''_F\right) + F''_{ZTF}\left(x''_F - e\right) \\ N''_{ZF} - N''_{ZWF} - F''_{YF} x''_F - F''_{YTF}\left(x''_F - e\right) \end{matrix} \right\| \tag{5-18}$$

where

$$z''_{TF} = h_0 \sec \theta_F + e \tan \theta_F \qquad (5\text{-}19)$$

and x''_F and z''_F are given in Equations 5-5 and 5-6.

Equation 1-11 is now written in matrix form for the front wheel and steering fork. This is the moment balance equation for the front wheel and steering fork about the axes of the coordinate system fixed in the steering fork.

$$\left\| I''_F \right\| \left\| \begin{matrix} \dot{p}'' \\ \dot{q}'' \\ \dot{r}'' + \ddot{\delta} \end{matrix} \right\| + \left\| \begin{matrix} p'' \\ q'' \\ r'' + \dot{\delta} \end{matrix} \right\| \times \left(\left\| I''_F \right\| \left\| \begin{matrix} p'' \\ q'' \\ r'' + \dot{\delta} \end{matrix} \right\| \right) = \left\| N''_F \right\| \qquad (5\text{-}20)$$

where

$$\left\| \begin{matrix} \dot{p}'' \\ \dot{q}'' \\ \dot{r}'' \end{matrix} \right\| = \left\| S^T \right\| \cdot \left\| \begin{matrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{matrix} \right\| \qquad (5\text{-}21)$$

and

$$\left\| \begin{matrix} p'' \\ q'' \\ r'' \end{matrix} \right\| = \left\| S^T \right\| \cdot \left\| \begin{matrix} p \\ q \\ r \end{matrix} \right\| \qquad (5\text{-}22)$$

$$\begin{Vmatrix} \dot{p}'' I''_{FX} - (\dot{r}'' + \ddot{\delta}) I''_{FXZ} - p'' q'' I''_{FXZ} - q''(r'' + \dot{\delta})(I''_{FY} - I''_{FZ}) \\ \dot{q}'' I''_{FY} + \left[p''^2 - (r'' + \dot{\delta})^2\right] I''_{FXZ} + p''(r'' + \dot{\delta})(I''_{FX} - I''_{FZ}) \\ -\dot{p}'' I''_{FXZ} + (\dot{r}'' + \ddot{\delta}) I''_{FZ} + q''(r'' + \dot{\delta}) I''_{FXZ} - p'' q''(I''_{FX} - I''_{FY}) \end{Vmatrix} = \begin{Vmatrix} N''_{XF} - N''_{XWF} + F''_{YF}\, z''_F - F''_{YTF}(z''_{TF} - z''_F) \\ N''_{YF} - N''_{YWF} - F''_{XF}\, z''_F + F''_{ZF}\, x''_F + F''_{XTF}(z''_{TF} - z''_F) + F''_{ZTF}(x''_F - e) \\ N''_{ZF} - N''_{ZWF} - F''_{YF}\, x''_F - F''_{YTF}(x''_F - e) \end{Vmatrix} \tag{5-23}$$

From Equation 5-23 the reaction moments N''_{XF} and N''_{YF} are written

$$\begin{aligned} N''_{XF} = {} & \dot{p}'' I''_{FX} - (\dot{r}'' + \ddot{\delta}) I''_{FXZ} - p'' q'' I''_{FXZ} - q''(r'' + \dot{\delta})(I''_{FY} - I''_{FZ}) \\ & + N''_{XWF} - F''_{YF}\, z''_F + F''_{YTF}(z''_{TF} - z''_F) \end{aligned} \tag{5-24}$$

$$\begin{aligned} N''_{YF} = {} & \dot{q}'' I''_{FY} + \left[p''^2 - (r'' + \dot{\delta})^2\right] I''_{FXZ} + p''(r'' + \dot{\delta})(I''_{FX} - I''_{FZ}) \\ & + N''_{YWF} + F''_{XF}\, z''_F - F''_{ZF}\, x''_F - F''_{XTF}(z''_{TF} - z''_F) - F''_{ZTF}(x''_F - e) \end{aligned} \tag{5-25}$$

The equation of motion of the steer degree of freedom of the front wheel and steering fork is now written from Equation 5-23.

$$\begin{aligned} & -\dot{p}'' I''_{FXZ} + (\dot{r}'' + \ddot{\delta}) I''_{FZ} + q''(r'' + \dot{\delta}) I''_{FXZ} - p'' q''(I''_{FX} - I''_{FY}) = \\ & N''_{ZF} - N''_{ZWF} - F''_{YF}\, x''_F - F''_{YTF}(x''_F - e) \end{aligned} \tag{5-26}$$

The vector moment acting on the front wheel and steering fork about the axes of the vehicle coordinate system is now found.

$$\begin{Vmatrix} N_{XF} \\ N_{YF} \\ N_{ZF} \end{Vmatrix} = \begin{Vmatrix} S \end{Vmatrix} \bullet \begin{Vmatrix} N''_{XF} \\ N''_{YF} \\ N''_{ZF} \end{Vmatrix} \tag{5-27}$$

$$\begin{aligned} N_{XF} = \cos\theta_F \Big[& \sin\theta_F \delta x''_F m_F \dot{u}_0 - z''_F m_F \dot{v}_0 + \cos\theta_F \delta x''_F m_F \dot{w}_0 \\ & + \left\{ \cos\theta_F I''_{FX} - \sin\theta_F I''_{FXZ} + z''_F m_F \right\} \dot{p} + \left\{ I''_{FX} - I''_{FY} - x''^2_F m_F \right\} \delta \dot{q} \\ & - \left\{ \sin\theta_F I''_{FX} + \cos\theta_F I''_{FXZ} + x_F z''_F m_F \right\} \dot{r} - \left\{ I''_{FXZ} + x''_F z''_F m_F \right\} \ddot{\delta} \\ & + \delta pr \left\{ \cos^2\theta_F - \sin^2\theta_F \right\} \left\{ I''_{FY} - I''_{FX} \right\} + \left\{ \delta \cos\theta_F \sin\theta_F \right\} \left\{ (p^2 - r^2)(I''_{FY} - I''_{FX}) - 2pr I''_{FXZ} \right\} \\ & + \left\{ \cos\theta_F p - \sin\theta_F r \right\} \left\{ \delta \dot{\delta} (I''_{FY} - I''_{FX}) - q I''_{FXZ} \right\} + \left\{ \sin\theta_F p + \cos\theta_F r \right\} \left\{ q (I''_{FZ} - I''_{FY}) + 2\delta\dot{\delta} I''_{FXZ} \right\} \\ & + q \dot{\delta} \left\{ I''_{FZ} - I''_{FY} \right\} + \left\{ \sin^2\theta_F p^2 + \cos^2\theta_F r^2 - q^2 + \dot{\delta}^2 \right\} \delta I''_{FXZ} \\ & - \left\{ \sin\theta_F p + \cos\theta_F r + \dot{\delta} \right\} \omega_F I_{WF} - \sin\theta_F \delta x''_F \gamma_{12} + z''_F \gamma_{22} - \cos\theta_F \delta x''_F \gamma_{32} \Big] \\ & + \left\{ h_0 + \sin\theta_F e \right\} F_{YTF} - \cos^2\theta_F \delta e F_{ZTF} + \sin\theta_F N''_{ZF} \end{aligned} \tag{5-28}$$

$$\begin{aligned} N_{YF} = \; & z_F m_F \dot{u}_0 - x_F m_F \dot{w}_0 + \left\{ \delta \cos\theta_F (I''_{FX} - I''_{FY}) - \delta \sin\theta_F I''_{FXZ} - \delta x''_F z_F m_F \right\} \dot{p} \\ & + \left\{ I''_{FY} + (x^2_F + z^2_F) m_F \right\} \dot{q} + \left\{ \delta \sin\theta_F (I''_{FY} - I''_{FX}) - \delta \cos\theta_F I''_{FXZ} + \delta x''_F z_F m_F \right\} \dot{r} \\ & - \left\{ I''_{FXZ} + (\sin^2\theta_F - \cos^2\theta_F) x_F z_F m_F - \cos\theta_F \sin\theta_F (x^2_F - z^2_F) m_F \right\} \delta \ddot{\delta} \\ & + \left\{ \cos^2\theta_F - \sin^2\theta_F \right\} \left\{ pr (I''_{FX} - I''_{FZ}) + (p^2 - r^2) I''_{FXZ} \right\} + \left\{ \cos\theta_F \sin\theta_F \right\} \Big\{ (p^2 - r^2)(I''_{FX} - I''_{FZ}) \\ & - 4pr I''_{FXZ} \Big\} + \left\{ \cos\theta_F p - \sin\theta_F r \right\} \left\{ \dot{\delta} (I''_{FX} - I''_{FZ}) + \delta q I''_{FXZ} \right\} \\ & + \left\{ \sin\theta_F p + \cos\theta_F r \right\} \left\{ \delta q (I''_{FX} - I''_{FY}) - 2\dot{\delta} I''_{FXZ} \right\} + \dot{\delta} \left\{ \delta q (I''_{FX} - I''_{FY}) - \dot{\delta} I''_{FXZ} \right\} \\ & - \left\{ \sin\theta_F p + \cos\theta_F r + \dot{\delta} \right\} \delta \omega_F I_{WF} - z_F \gamma_{12} + x_F \gamma_{32} - h_0 F_{XTF} + \ell_F F_{ZTF} \end{aligned} \tag{5-29}$$

$$
\begin{aligned}
N_{ZF} = & -\sin\theta_F \Big[\sin\theta_F\, \delta\, x_F''\, m_F\, \dot{u}_0 - z_F''\, m_F\, \dot{v}_0 + \cos\theta_F\, \delta\, x_F''\, m_F\, \dot{w}_0 \\
& + \left\{ \cos\theta_F\, I_{FX}'' - \sin\theta_F\, I_{FXZ}'' + z_F''\, z_F\, m_F \right\} \dot{p} + \left\{ I_{FX}'' - I_{FY}'' - x_F''^{\,2}\, m_F \right\} \delta \dot{q} \\
& - \left\{ \sin\theta_F\, I_{FX}'' + \cos\theta_F\, I_{FXZ}'' + x_F\, z_F''\, m_F \right\} \dot{r} - \left\{ I_{FXZ}'' + x_F''\, z_F''\, m_F \right\} \ddot{\delta} \\
& + \delta p r \left\{ \cos^2\theta_F - \sin^2\theta_F \right\} \left\{ I_{FY}'' - I_{FX}'' \right\} + \left\{ \delta \cos\theta_F \sin\theta_F \right\} \left\{ (p^2 - r^2)(I_{FY}'' - I_{FX}'') - 2 p r I_{FXZ}'' \right\} \\
& + \left\{ \cos\theta_F\, p - \sin\theta_F\, r \right\} \left\{ \delta \dot{\delta} (I_{FY}'' - I_{FX}'') - q I_{FXZ}'' \right\} + \left\{ \sin\theta_F\, p + \cos\theta_F\, r \right\} \left\{ q (I_{FZ}'' - I_{FY}'') + 2 \delta \dot{\delta} I_{FXZ}'' \right. \\
& + q \dot{\delta} \left\{ I_{FZ}'' - I_{FY}'' \right\} + \left\{ \sin^2\theta_F\, p^2 + \cos^2\theta_F\, r^2 - q^2 + \dot{\delta}^2 \right\} \delta I_{FXZ}'' \\
& - \left\{ \sin\theta_F\, p + \cos\theta_F\, r + \dot{\delta} \right\} \text{[illegible]} - \sin\theta_F\, \delta\, x_F''\, \gamma_{12} + z_F''\, \gamma_{22} - \cos\theta_F\, \delta\, x_F''\, \gamma_{32} \Big] \\
& - \tan\theta_F \left\{ h_0 + \sin\theta_F\, e \right\} F_{YTF} + \sin^2\theta_F\, \delta\, e\, F_{XTF} + \sin\theta_F \cos\theta_F\, \delta\, e\, F_{ZTF} \\
& + \cos\theta_F\, N_{ZF}''
\end{aligned}
\qquad (5\text{-}30)
$$

where: $\ell_F = \tan\theta_F\, h_0 + \sec\theta_F\, e$ (5-31)

Section 6

FORCE AND MOMENT EQUATION FOR REAR WHEEL AND FRAME

Equation 1-7 is now written for the inertial acceleration of the c.g. of the rear wheel ~~and frame~~ in the vehicle coordinate system.

$$\left\| \begin{matrix} \dot{u}_R \\ \dot{v}_R \\ \dot{w}_R \end{matrix} \right\| = \left\| \begin{matrix} \dot{u}_0 + q w_0 - r v_0 \\ \dot{v}_0 - p w_0 + r u_0 \\ \dot{w}_0 + p v_0 - q u_0 \end{matrix} \right\| + \left\| \begin{matrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{matrix} \right\| \times \left\| \begin{matrix} x_R \\ 0 \\ z_R \end{matrix} \right\| + \left\| \begin{matrix} p \\ q \\ r \end{matrix} \right\| \times \left\| \begin{matrix} p \\ q \\ r \end{matrix} \right\| \times \left\| \begin{matrix} x_R \\ 0 \\ z_R \end{matrix} \right\| \qquad (6\text{-}1)$$

The components of the acceleration are now written in expanded form.

$$\dot{u}_R = \dot{u}_0 + q w_0 - r v_0 + \dot{q} z_R - (q^2 + r^2) x_R + p r z_R \qquad (6\text{-}2)$$

$$\dot{v}_R = \dot{v}_0 - p w_0 + r u_0 - \dot{p} z_R + \dot{r} x_R + p q x_R + q r z_R \qquad (6\text{-}3)$$

$$\dot{w}_R = \dot{w}_0 + p v_0 - q u_0 - \dot{q} x_R + p r x_R - (p^2 + q^2) z_R \qquad (6\text{-}4)$$

The force balance equations for the rear wheel and frame in the X, Y, Z directions are given below.

$$m_R \dot{u}_R = F_{XTR} - F_{XF} - F_{XD} - m_R g \sin\theta \qquad (6\text{-}5)$$

$$m_R \dot{v}_R = F_{YTR} - F_{YF} - F_{YD} + m_R g \cos\theta \sin\phi \qquad (6\text{-}6)$$

$$m_R \dot{w}_R = F_{ZTR} - F_{ZF} - F_{ZD} + m_R g \cos\theta \cos\phi \qquad (6\text{-}7)$$

where:

$F_{XTR}, F_{YTR}, F_{ZTR}$ are the X, Y, Z components of the tire forces acting on the rear wheel at the ground contact point.

F_{XF}, F_{YF}, F_{ZF} are the X, Y, Z components of the reaction force acting on the steering fork at the origin of the vehicle coordinate system.

F_{XD}, F_{YD}, F_{ZD} are the X, Y, Z components of the reaction force acting on the driver at the driver roll center.

The equations of translational motion of the vehicle in the X, Y, Z directions are as follows.

<u>In the X direction</u>

$$\Sigma M \dot{u}_0 + \gamma_3 \dot{q} - \gamma_2 \dot{r} - m_F \cos\theta_F \delta z_F \ddot{\delta} = \gamma_{11} + \gamma_{12} + \gamma_{13} + F_{XTR} + F_{XTF} \quad (6\text{-}8)$$

<u>In the Y direction</u>

$$\Sigma M \dot{v}_0 + \gamma_1 \dot{r} - \gamma_3 \dot{p} + m_F z_F \ddot{\delta} - m_D \rho_D \ddot{\phi}_D = \gamma_{21} + \gamma_{22} + \gamma_{23} + F_{YTR} + F_{YTF} \quad (6\text{-}9)$$

<u>In the Z direction</u>

$$\Sigma M \dot{w}_0 - \gamma_1 \dot{q} + \gamma_2 \dot{p} - m_D \rho_D \phi_D \ddot{\phi}_D = \gamma_{31} + \gamma_{32} + \gamma_{33} + F_{ZTR} + F_{ZTF} \quad (6\text{-}10)$$

Where ΣM and the γ terms are defined in Equations 7-1 through 7-13.

The moment on the rear wheel which reacts the gyroscopic effect of the spinning wheel is given below in matrix form. It is assumed that the moment of inertia about a diameter through the wheel is equal to one half the moment of inertia about the spin axis, I_{WR}.

$$\begin{Vmatrix} N_{XWR} \\ N_{YWR} \\ N_{ZWR} \end{Vmatrix} = \begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \times \left(\begin{Vmatrix} \frac{I_{WR}}{2} & 0 & 0 \\ 0 & I_{WR} & 0 \\ 0 & 0 & \frac{I_{WR}}{2} \end{Vmatrix} \begin{Vmatrix} p \\ q + \omega_R \\ r \end{Vmatrix} \right) \qquad (6\text{-}11)$$

where

N_{XWR}, N_{YWR}, N_{ZWR} are the X, Y, Z components of the moment on the rear wheel which reacts the gyroscopic effect.

ω_R is the spin velocity of the rear wheel (assumed to be the forward velocity of the bicycle divided by the rolling radius of the wheel).

I_{WR} is the moment of inertia of the rear wheel about its spin axis.

Assuming that the pitch velocity, q, is negligible compared to the wheel spin velocity, ω_R, Equation 6-11 reduces to the following.

$$N_{XWR} = -r\omega_R I_{WR} \qquad (6\text{-}12)$$

$$N_{YWR} = 0 \qquad (6\text{-}13)$$

$$N_{ZWR} = p\omega_R I_{WR} \qquad (6\text{-}14)$$

Equation 1-11 is now written in matrix form for the rear wheel and frame. This is the moment balance equation for the rear wheel and frame about the origin of the vehicle coordinate system.

$$\|I_R\| \begin{Vmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Vmatrix} + \begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \times \left(\|I_R\| \begin{Vmatrix} p \\ q \\ r \end{Vmatrix} \right) = \|N_R\| \tag{6-15}$$

$\|I_R\|$ is the inertia tensor of the rear wheel and frame about the origin of the vehicle coordinate system.

$$\|I_R\| = \begin{Vmatrix} I_{RX} + m_R z_R^2 & 0 & -I_{RXZ} - m_R x_R z_R \\ 0 & I_{RY} + m_R (x_R^2 + z_R^2) & 0 \\ -I_{RXZ} - m_R x_R z_R & 0 & I_{RZ} + m_R x_R^2 \end{Vmatrix} \tag{6-16}$$

where:

I_{RX}, I_{RY}, I_{RZ} are the X, Y, Z moments of inertia of the rear wheel and frame about its c. g.

I_{RXZ} is the X-Z product of inertia of the rear wheel and frame about its c.g. (Since the vehicle is assumed to be symmetrical about the X-Z plane, its X-Y and Y-Z products of inertia are zero.

$\|N_R\|$ is the vector moment acting on the rear wheel and frame at the origin of the vehicle coordinate system.

$$\|N_R\| = \begin{Vmatrix} -N_{XF} - N_{XD} - N_{XWR} + (z_D - \rho_D) F_{YD} - h_0 F_{YTR} + m_R z_R (\dot{v}_0 - p w_0 + r u_0 - g \cos\theta \sin\phi) \\ -N_{YF} - N_{YD} - N_{YWR} - (z_D - \rho_D) F_{XD} + x_D F_{ZD} + h_0 F_{XTR} - \ell_R F_{ZTR} \\ + m_R x_R (\dot{w}_0 + p v_0 - q u_0 - g \cos\theta \cos\phi) - m_R z_R (\dot{u}_0 + q w_0 - r v_0 + g \sin\theta) \\ -N_{ZF} - N_{ZD} - N_{ZWR} - x_D F_{YD} + \ell_R F_{YTR} - m_R x_R (\dot{v}_0 - p w_0 + r u_0 - g \cos\theta \sin\phi) \end{Vmatrix} \tag{6-17}$$

Equation 6-15 is now written as follows.

$$\left\| \begin{matrix} \dot{p}(I_{RX}+m_R z_R^2)-(\dot{r}+pq)(I_{RXZ}+m_R x_R z_R)+qr(I_{RZ}-I_{RY}-m_R z_R^2) \\ \dot{q}\left(I_{RY}+m_R\left\{x_R^2+z_R^2\right\}\right)+(p^2-r^2)(I_{RXZ}+m_R x_R z_R)+pr\left(I_{RX}-I_{RZ}-m_R\left\{x_R^2-z_R^2\right\}\right) \\ \dot{r}(I_{RZ}+m_R x_R^2)-(\dot{p}-qr)(I_{RXZ}+m_R x_R z_R)+pq(I_{RY}-I_{RX}+m_R x_R^2) \end{matrix} \right\| = \left\| N_R \right\| \tag{6-18}$$

<u>equation of motion about the X axis</u>

$$\begin{aligned} &\delta \cos\theta_F \sin\theta_F x_F'' m_F \dot{u}_0 - \left\{z_R m_R + (z_D-\rho_D) m_D + \cos\theta_F z_F'' m_F\right\} \dot{v}_0 \\ &+ \delta \cos^2\theta_F x_F'' m_F \dot{w}_0 + \Big\{I_{RX} + z_R^2 m_R + z_D(z_D-\rho_D) m_D + \cos^2\theta_F I_{FX}'' \\ &- \cos\theta_F \sin\theta_F I_{FXZ}'' + \cos\theta_F z_F'' z_F m_F\Big\} \dot{p} + \left\{I_{FX}'' - I_{FY}'' - x_F''^2 m_F\right\} \delta \dot{q} \\ &- \Big\{I_{RXZ} + x_R z_R m_R + x_D(z_D-\rho_D) m_D + \cos\theta_F \sin\theta_F I_{FX}'' + \cos^2\theta_F I_{FXZ}'' \\ &+ \cos\theta_F x_F z_F'' m_F\Big\} \dot{r} - \left\{\cos\theta_F I_{FXZ}'' + \cos\theta_F x_F'' z_F'' m_F\right\} \ddot{\delta} + \rho_D\left\{z_D-\rho_D\right\} m_D \ddot{\phi}_D \\ &\quad = pq I_{RXZ} + qr(I_{RY}-I_{RZ}) - z_R \gamma_{21} - (z_D-\rho_D)\gamma_{23} - r\omega_R I_{WR} \\ &\quad - \cos\theta_F\Big[\delta pr\left\{\cos^2\theta_F - \sin^2\theta_F\right\}\left\{I_{FY}''-I_{FX}''\right\} + \left\{\delta\cos\theta_F\sin\theta_F\right\}\left\{(p^2-r^2)(I_{FY}''-I_{FX}'') - 2pr I_{FXZ}''\right\} \\ &\quad + \left\{\cos\theta_F p - \sin\theta_F r\right\}\left\{\delta\dot{\delta}(I_{FY}''-I_{FX}'') - q I_{FXZ}''\right\} + \left\{\sin\theta_F p + \cos\theta_F \quad\right\}\left\{q(I_{FZ}''-I_{FY}'') + 2\delta\dot{\delta} I_{FXZ}''\right\} \\ &\quad + q\dot{\delta}\left\{I_{FZ}''-I_{FY}''\right\} + \left\{\sin^2\theta_F p^2 + \cos^2\theta_F r^2 - q^2 + \dot{\delta}^2\right\} \delta I_{FXZ}'' \\ &\quad - \left\{\sin\theta_F p + \cos\theta_F r + \dot{\delta}\right\} \omega_F I_{WF} - \sin\theta_F \delta x_F'' \gamma_{12} + z_F'' \gamma_{22} - \cos\theta_F \delta x_F'' \gamma_{32}\Big] \\ &\quad - \left\{h_0 + \sin\theta_F e\right\} F_{YTF} - h_0 F_{YTR} + \cos^2\theta_F \delta e F_{ZTF} - \sin\theta_F N_{ZF}'' \end{aligned} \tag{6-19}$$

equation of motion about the Y axis

$$
\begin{aligned}
&\gamma_3 \dot{u}_0 - \gamma_1 \dot{w}_0 + \Big[\delta\big\{\cos\theta_F(I''_{FX}-I''_{FY}) - \sin\theta_F\, I''_{FXZ} - x''_F z_F m_F\big\} \\
&+ \phi_D\big\{I_{DXZ} + \rho_D x_D m_D\big\}\Big]\dot{p} + \big\{I''_{FY} + (x_F^2 + z_F^2) m_F + I_{DY} + (x_D^2 + z_D^2) m_D \\
&+ I_{RY} + (x_R^2 + z_R^2) m_R\big\}\dot{q} + \Big[\delta\big\{\sin\theta_F(I''_{FY}-I''_{FX}) - \cos\theta_F\, I''_{FXZ} \\
&+ x''_F z_F m_F\big\} + \phi_D\big\{I_{DY} - I_{DZ} + \rho_D z_D m_D\big\}\Big]\dot{r} \quad \delta\big\{I''_{FXZ} + (\sin^2\theta_F - \cos^2\theta_F) \\
&x_F z_F m_F - \cos\theta_F \sin\theta_F (x_F^2 - z_F^2) m_F\big\}\ddot{\delta} + \phi_D\big\{I_{DXZ} + \rho_D x_D m_D\big\}\ddot{\phi}_D \\
&\quad = -(p^2 - r^2) I_{RXZ} - pr(I_{RX} - I_{RZ}) - \big\{(p + \dot{\phi}_D)^2 + \phi_D q r - r^2\big\} I_{DXZ} \\
&\qquad - r(p + \dot{\phi}_D)(I_{DX} - I_{DZ}) + \phi_D q (p + \dot{\phi}_D)(I_{DY} - I_{DZ}) \\
&\qquad - \big\{\cos^2\theta_F - \sin^2\theta_F\big\}\big\{pr(I''_{FX} - I''_{FZ}) + (p^2 - r^2) I''_{FXZ}\big\} - \big\{\cos\theta_F \sin\theta_F\big\}\big\{(p^2 - r^2)(I''_{FX} - I''_{FZ}) \\
&\qquad - 4pr\, I''_{FXZ}\big\} - \big\{\cos\theta_F\, p - \sin\theta_F\, r\big\}\big\{\dot{\delta}(I''_{FX} - I''_{FZ}) + \delta q\, I''_{FXZ}\big\} \\
&\qquad - \big\{\sin\theta_F\, p + \cos\theta_F\, r\big\}\big\{\delta q\,(I''_{FX} - I''_{FY}) - 2\dot{\delta} I''_{FXZ}\big\} - \dot{\delta}\big\{\delta q\,(I''_{FX} - I''_{FY}) \\
&\qquad - \dot{\delta} I''_{FXZ}\big\} + \big\{\sin\theta_F\, p + \cos\theta_F\, r + \dot{\delta}\big\}\delta\, \omega_F I_{WF} \\
&\qquad + z_R \gamma_{11} + z_F \gamma_{12} + z_D \gamma_{13} - x_R \gamma_{31} - x_F \gamma_{32} - x_D \gamma_{33} \\
&\qquad + h_0(F_{XTR} + F_{XTF}) - \ell_R F_{ZTR} - \ell_F F_{ZTF}
\end{aligned}
\qquad (6\text{-}20)
$$

equation of motion about the Z axis

$$
\begin{aligned}
&-(\delta \sin^2\theta_F\, x''_F m_F + \phi_D \rho_D m_D)\dot{u}_0 + (x_R m_R + x_D m_D + \sin\theta_F\, z''_F m_F)\dot{v}_0 \\
&-\delta \sin\theta_F \cos\theta_F\, x''_F m_F \dot{w}_0 - \Big\{ I_{RXZ} + x_R z_R m_R + I_{DXZ} + x_D z_D m_D \\
&+ \sin\theta_F(\cos\theta_F I''_{FX} - \sin\theta_F I''_{FXZ} + z''_F z_F m_F)\Big\}\dot{p} + \Big\{\phi_D(I_{DY} - I_{DZ} \\
&+ \rho_D z_D m_D) - \delta\sin\theta_F(I''_{FX} - I''_{FY} - x''^2_F m_F)\Big\}\dot{q} + \Big\{I_{RZ} + x_R^2 m_R \\
&+ I_{DZ} + x_D^2 m_D + \sin\theta_F(\sin\theta_F I''_{FX} + \cos\theta_F I''_{FXZ} + x_F z''_F m_F)\Big\}\dot{r} \\
&+ \sin\theta_F(I''_{FXZ} + x''_F z''_F m_F)\ddot{\delta} - (I_{DXZ} + \rho_D x_D m_D)\ddot{\phi}_D \\
&\quad = pq(I_{RX} - I_{RY}) - qr I_{RXZ} + q(p + \dot{\phi}_D)(I_{DX} - I_{DY}) \\
&\quad -\phi_D r(p + \dot{\phi}_D)(I_{DY} - I_{DZ}) - \left\{qr + \phi_D(p + \dot{\phi}_D)^2 - \phi_D q^2\right\} I_{DXZ} \\
&\quad + \sin\theta_F\Big[\delta pr\left\{\cos^2\theta_F - \sin^2\theta_F\right\}\left\{I''_{FY} - I''_{FX}\right\} + \left\{\delta\cos\theta_F\sin\theta_F\right\} \\
&\quad \left\{(p^2 - r^2)(I''_{FY} - I''_{FX}) - 2pr I''_{FXZ}\right\} + \left\{\cos\theta_F p - \sin\theta_F r\right\}\Big\{\delta\dot{\delta}(I''_{FY} - I''_{FX}) \\
&\quad - q I''_{FXZ}\Big\} + \left\{\sin\theta_F p + \cos\theta_F r\right\}\left\{q(I''_{FZ} - I''_{FY}) + 2\delta\dot{\delta} I''_{FXZ}\right\} \\
&\quad + q\dot{\delta}\left\{I''_{FZ} - I''_{FY}\right\} + \left\{\sin^2\theta_F p^2 + \cos^2\theta_F r^2 - q^2 + \dot{\delta}^2\right\}\delta I''_{FXZ} \\
&\quad - \left\{\sin\theta_F p + \cos\theta_F r + \dot{\delta}\right\}\omega_F I_{WF} - \sin\theta_F\delta x''_F \gamma_{12} + z''_F \gamma_{22} \\
&\quad - \cos\theta_F \delta x''_F \gamma_{32}\Big] - p\omega_R I_{WR} + x_D\gamma_{23} + x_R\gamma_{21} + \phi_D\rho_D\gamma_{13} \\
&\quad + l_R F_{YTR} - \tan\theta_F\left\{h_0 + \sin\theta_F e\right\} F_{YTF} + \sin^2\theta_F \delta e F_{XTF} \\
&\quad + \sin\theta_F\cos\theta_F \delta e F_{ZTF} + \cos\theta_F N''_{ZF} \qquad (6\text{-}21)
\end{aligned}
$$

Section 7
MATRIX EQUATION OF MOTION

The following terms are defined to permit simplification of the equations of motion.

$$\Sigma M = m_R + m_F + m_D \tag{7-1}$$

$$\delta_1 = m_R x_R + m_F x_F + m_D x_D \tag{7-2}$$

$$\delta_2 = m_F \cos\theta_F \,\delta x_F - m_D \rho_D \phi_D \tag{7-3}$$

$$\delta_3 = m_R z_R + m_F z_F + m_D z_D \tag{7-4}$$

$$\delta_{11} = M_R\left(-q w_0 + r v_0 - g \sin\theta + x_R\left\{q^2 + r^2\right\} - z_R\, pr\right) \tag{7-5}$$

$$\delta_{21} = M_R\left(p w_0 - r u_0 + g \cos\theta \sin\phi - x_R\, pq - z_R\, qr\right) \tag{7-6}$$

$$\delta_{31} = M_R\left(-p v_0 + q u_0 + g \cos\theta - x_R\, pr + z_R\left\{p^2 + q^2\right\}\right) \tag{7-7}$$

$$\begin{aligned}\delta_{13} = M_D\Big(&-q w_0 + r v_0 - g \sin\theta + x_D\left\{q^2 + r^2\right\} + \left\{\rho_D \phi_D\right\} pq - z_D\, pr \\ &+ \left\{2\rho_D \phi_D\right\} q\dot{\phi}_D - 2\rho_D\, r\dot{\phi}_D\Big)\end{aligned} \tag{7-8}$$

$$\begin{aligned}\delta_{23} = M_D\Big(&p w_0 - r u_0 + g \cos\theta \sin\phi - \left\{\rho_D \phi_D\right\}\left\{p^2 + r^2\right\} - x_D\, pq \\ &- z_D\, qr - \left\{2\rho_D \phi_D\right\} p\dot{\phi}_D - \left\{\rho_D \phi_D\right\} \dot{\phi}_D^2\Big)\end{aligned} \tag{7-9}$$

$$\begin{aligned}\delta_{33} = M_D\Big(&-p v_0 + q u_0 + g \cos\theta \cos\phi - x_D\, pr + \left\{\rho_D \phi_D\right\} qr + z_D\left\{p^2 + q^2\right\} \\ &+ 2\rho_D\, p\dot{\phi}_D + \rho_D \dot{\phi}_D^2\Big)\end{aligned} \tag{7-10}$$

$$
\begin{aligned}
\gamma_{1z} = m_F \Big(& -q w_0 + r v_0 - g \sin\theta + z_F \left\{ q^2 + r^2 \right\} - x_F'' \delta p q \\
& - z_F p r - z_F \cos\theta \, p \dot{\delta} + \left\{ x_F'' + z_F \cos\theta_F \right\} r \dot{\delta} \\
& + x_F'' \cos\theta_F \dot{\delta}^2 - x_F'' \delta \sin\theta_F q \dot{\delta} \Big)
\end{aligned}
\tag{7-11}
$$

$$
\begin{aligned}
\gamma_{2z} = m_F \Big(& p w_0 - r u_0 + g \cos\theta \sin\phi - z_F p q + x_F'' \delta \left\{ p^2 + r^2 \right\} \\
& - z_F q r - z_F'' q \dot{\delta} + 2 x_F'' \delta \cos\theta_F r \dot{\delta} + 2 x_F'' \delta \sin\theta_F p \dot{\delta} \\
& + x_F'' \delta \dot{\delta}^2 \Big)
\end{aligned}
\tag{7-12}
$$

$$
\begin{aligned}
\gamma_{3z} = m_F \Big(& - p v_0 + q u_0 + g \cos\theta \cos\phi - z_F p r - x_F'' \delta q r \\
& + z_F \;\; p^2 + q^2 \; - x_F'' - z_F \sin\theta_F \;\; p \dot{\delta} - x_F'' \delta \cos\theta_F q \dot{\delta} \\
& - z_F \sin\theta_F r \dot{\delta} - x_F'' \sin\theta_F \dot{\delta}^2 \Big)
\end{aligned}
\tag{7-13}
$$

ΣM	0	0	0	γ_3	$-\gamma_2$	$-\delta\cos\theta_F x_F m_F$
0	ΣM	0	$-\gamma_3$	0	γ_1	$x_F m_F$
0	0	ΣM	γ_2	$-\gamma_1$	0	0
$\delta\cos\theta_F\sin\theta_F x_F'' m_F$	$-z_R m_R-(z_D-\rho_D)m_D$ $-\cos\theta_F z_F'' m_F$	$\delta\cos^2\theta_F x_F'' m_F$	$I_{RX}+z_R^2 m_R+z_D(z_D-\rho_D)m_D$ $+\cos^2\theta_F I_{FX}''-\cos\theta_F\sin\theta_F I_{FXZ}''$ $+\cos\theta_F z_F'' z_F m_F$	$\delta(I_{FX}''-I_{FY}''-x_F''^2 m_F)$	$-I_{RXZ}-x_R z_R m_R-x_D(z_D-\rho_D)m_D$ $-\cos\theta_F\sin\theta_F I_{FX}''-\cos^2\theta_F I_{FXZ}''$ $-\cos\theta_F x_F z_F'' m_F$	$-\cos\theta_F I_{FXZ}''$ $-\cos\theta_F x_F'' z_F'' m_F$
γ_3	0	$-\gamma_1$	$\delta\{\cos\theta_F(I_{FX}''-I_{FY}'')-\sin\theta_F I_{FXZ}''$ $-x_F'' x_F m_F\}+\phi_D\{I_{DXZ}+\rho_D x_D m_D\}$	$I_{FY}''+(x_F^2+z_F^2)m_F+I_{DY}$ $+(x_D^2+z_D^2)m_D+I_{RY}+(x_R^2+z_R^2)m_R$	$\delta\{\sin\theta_F(I_{FY}''-I_{FX}'')-\cos\theta_F I_{FXZ}''$ $+x_F'' z_F m_F\}+\phi_D\{I_{DY}-I_{DZ}+\rho_D z_D m_D\}$	$-\delta\{I_{FXZ}''+(\sin^2\theta_F$ $-\cos^2\theta_F)x_F z_F m_F$ $-\cos\theta_F\sin\theta_F(x_F^2$ $-z_F^2)m_F$
$-\delta\sin^2\theta_F x_F'' m_F$ $-\phi_D\rho_D m_D$	$x_R m_R+x_D m_D$ $+\sin\theta_F z_F'' m_F$	$-\delta\sin\theta_F\cos\theta_F x_F'' m_F$	$-I_{RXZ}-x_R z_R m_R-I_{DXZ}$ $-x_D z_D m_D-\sin\theta_F\{\cos\theta_F I_{FX}''$ $-\sin\theta_F I_{FXZ}''+z_F'' z_F m_F\}$	$\phi_D(I_{DY}-I_{DZ}+\rho_D z_D m_D)$ $-\delta\sin\theta_F(I_{FX}''-I_{FY}''-x_F''^2 m_F)$	$I_{RZ}+x_R^2 m_R+I_{DZ}+x_D^2 m_D$ $+\sin\theta_F(\sin\theta_F I_{FX}''+\cos\theta_F$ $I_{FXZ}''+x_F z_F'' m_F)$	$\sin\theta_F I_{FXZ}''$ $+\sin\theta_F x_F'' z_F'' m_F$
$-\delta\cos\theta_F x_F'' m_F$	$x_F'' m_F$	$\delta\sin\theta_F x_F'' m_F$	$\sin\theta_F I_{FZ}''-\cos\theta_F I_{FXZ}''-x_F'' z_F m_F$	$-\delta(I_{FXZ}''+x_F'' z_F'' m_F)$	$\cos\theta_F I_{FZ}''+x_F'' x_F m_F$ $+\sin\theta_F I_{FXZ}''$	$I_{FZ}''+x_F''^2 m_F$
0	$-\rho_D m_D$	$-\phi_D\rho_D m_D$	$I_{DX}+\rho_D z_D m_D$	$\phi_D(I_{DXZ}+\rho_D x_D m_D)$	$-I_{DXZ}-\rho_D x_D m_D$	0

$-\delta\cos\theta_F x_F m_F$	0	$\dot{u}_0$
$x_F m_F$	$-\rho_D m_D$	$\dot{v}_0$
0	$-\phi_D\rho_D m_D$	$\dot{w}_0$
$-\cos\theta_F I''_{FXZ}$ $-\cos\theta_F x''_F z''_F m_F$	$\rho_D\{z_D-\rho_D\}m_D$	$\dot{p}$
$-\delta\{I''_{FXZ}+(\sin^2\theta_F$ $-\cos^2\theta_F)x_F z_F m_F$ $-\cos\theta_F\sin\theta_F(x_F^2$ $-z_F^2)m_F$	$\phi_D I_{DXZ}$ $-\phi_D\rho_D x_D m_D$	$\dot{q}$
$\sin\theta_F I''_{FXZ}$ $+\sin\theta_F x''_F z''_F m_F$	$-I_{DXZ}$ $-\rho_D x_D m_D$	$\dot{r}$
$I''_{FX}+x''^2_F m_F$	0	$\ddot{\delta}$
0	$I_{DX}+\rho_D^2 m_D$	$\ddot{\phi}_D$

$=$

$$\gamma_{11}+\gamma_{12}+\gamma_{13}+F_{XTR}+F_{XTF}$$

$$\gamma_{21}+\gamma_{22}+\gamma_{23}+F_{YTR}+F_{YTF}$$

$$\gamma_{31}+\gamma_{32}+\gamma_{33}+F_{ZTR}+F_{ZTF}$$

$$\begin{aligned}
&pqI_{RXZ}+qr(I_{RY}-I_{RZ})-z_R\gamma_{21}-(z_D-\rho_D)\gamma_{23}+r\omega_R I_{WR}-\cos\theta_F\Big[\delta pr\{\cos^2\theta_F-\sin^2\theta_F\}\\
&\{I''_{FY}-I''_{FX}\}+\{\delta\cos\theta_F\sin\theta_F\}\{(p^2-r^2)(I''_{FY}-I''_{FX})-2prI''_{FXZ}\}+\{\cos\theta_F p-\sin\theta_F r\}\{\delta\dot{\delta}(I''_{FY}-I''_{FX})\\
&-qI''_{FXZ}\}+\{\sin\theta_F p+\cos\theta_F r\}\{q(I''_{FZ}-I''_{FY})+2\delta\dot{\delta}I''_{FXZ}\}+q\dot{\delta}\{I''_{FZ}-I''_{FY}\}+\{\sin^2\theta_F p^2+\cos^2\theta_F r^2\\
&-p^2+\dot{\delta}^2\}\delta I''_{FXZ}-\{\sin\theta_F p+\cos\theta_F r+\dot{\delta}\}\omega_F I_{WF}-\sin\theta_F\delta x''_F\gamma_{12}+z''_F\gamma_{22}-\cos\theta_F\delta x''_F\gamma_{32}\Big]\\
&-\{h_0+\sin\theta_F e\}F_{YTF}-h_0F_{YTR}+\cos^2\theta_F\delta eF_{ZTF}-\sin\theta_F N''_{ZF}
\end{aligned}$$

$$\begin{aligned}
&-(p^2-r^2)I_{RXZ}-pr(I_{RX}-I_{RZ})-\{(p+\dot{\phi}_D)^2+\phi_D qr-r^2\}I_{DXZ}-r(p+\dot{\phi}_D)(I_{DX}-I_{DZ})\\
&+\phi_D q(p+\dot{\phi}_D)(I_{DY}-I_{DZ})-\{\cos^2\theta_F-\sin^2\theta_F\}\{pr(I''_{FX}-I''_{FZ})+(p^2-r^2)I''_{FXZ}\}-\{\cos\theta_F\sin\theta_F\}\\
&\{(p^2-r^2)(I''_{FX}-I''_{FZ})-4prI''_{FXZ}\}+\{\cos\theta_F p-\sin\theta_F r\}\{\dot{\delta}(I''_{FX}-I''_{FZ})+\delta qI''_{FXZ}\}-\{\sin\theta_F p\\
&+\cos\theta_F r\}\{\delta q(I''_{FX}-I''_{FY})-2\dot{\delta}I''_{FXZ}\}-\dot{\delta}\{\delta q(I''_{FX}-I''_{FY})-\dot{\delta}I''_{FXZ}\}+\{\sin\theta_F p+\cos\theta_F r\\
&+\dot{\delta}\}\delta\omega_F I_{WF}+z_R\gamma_{11}+z_F\gamma_{12}+z_D\gamma_{13}-x_R\gamma_{31}-x_F\gamma_{32}-x_D\gamma_{33}+h_0(F_{XTR}+F_{XTF})-l_RF_{ZTR}-l_FF_{ZTF}
\end{aligned}$$

$$\begin{aligned}
&pq(I_{RX}-I_{RY})-qrI_{RXZ}+q(p+\dot{\phi}_D)(I_{DX}-I_{DY})-\phi_D r(p+\dot{\phi}_D)(I_{DY}-I_{DZ})-\{qr+\phi_D(p+\dot{\phi}_D)^2-\phi_D q^2\}\\
&I_{DXZ}+\sin\theta_F\Big[\delta pr\{\cos^2\theta_F-\sin^2\theta_F\}\{I''_{FY}-I''_{FX}\}+\{\delta\cos\theta_F\sin\theta_F\}\{(p^2-r^2)(I''_{FY}-I''_{FX})-2prI''_{FXZ}\}\\
&+\{\cos\theta_F p-\sin\theta_F r\}\{\delta\dot{\delta}(I''_{FY}-I''_{FX})-qI''_{FXZ}\}+\{\sin\theta_F p+\cos\theta_F r\}\{q(I''_{FZ}-I''_{FY})\\
&-2\delta\dot{\delta}I''_{FXZ}\}+q\dot{\delta}\{I''_{FZ}-I''_{FY}\}+\{\sin^2\theta_F p^2+\cos^2\theta_F r^2-q^2+\dot{\delta}^2\}\delta I''_{FXZ}-\{\sin\theta_F p+\cos\theta_F r\\
&+\dot{\delta}\}\omega_F I_{WF}-\sin\theta_F\delta x''_F\gamma_{12}+z''_F\gamma_{22}-\cos\theta_F\delta x''_F\gamma_{32}\Big]-p\omega_R I_{WR}+x_D\gamma_{23}+x_R\gamma_{21}+\phi_D\rho_D\gamma_{13}\\
&+l_RF_{YTR}-\tan\theta_F\{h_0+\sin\theta_F e\}F_{YTF}+\sin^2\theta_F\delta eF_{XTF}+\sin\theta_F\cos\theta_F\delta eF_{ZTF}+\cos\theta_F N''_{ZF}
\end{aligned}$$

$$\begin{aligned}
&\{-\delta\cos\theta_F p+q+\delta\sin\theta_F r\}(\sin\theta_F p+\cos\theta_F r+\dot{\delta})I''_{FXZ}-(\cos\theta_F p+\delta q-\sin\theta_F r)(I''_{FX}-I''_{FY})\}\\
&-(\cos\theta_F p+\delta q-\sin\theta_F r)\omega_F I_{WF}-\delta\cos\theta_F x''_F\gamma_{12}+x''_F\gamma_{22}+\delta\sin\theta_F x''_F\gamma_{32}+eF''_{YTF}+N''_{ZF}
\end{aligned}$$

$$(q+\phi_D r)(p+\dot{\phi}_D)I_{DXZ}+(qr-\phi_D\{q^2-r^2\})(I_{DY}-I_{DZ})-\rho_D\gamma_{23}-\phi_D\rho_D\gamma_{33}+N_{XD}$$

B