In our comparisons, we found that in fewer than half of the papers do the equations of motion resemble those derived in Chapter III. In fact, of the papers discussed, we found that only two derived fully general and perfectly correct results' (one of these was later employed by another investigator). Several more were either a little less general or had minor errors which an alert reader might catch. A number of others were too complicated to check in full (but some of them raised some questions we could not answer). Finally, several are just plain wrong.

Results of Chronological Comparison of Linearized Equations of Motion[3]

Whipple, 1899

The first to formally derive a fully general and scholarly set of equations for the Basic bicycle model was Whipple in 1899. He treats the front and the rear parts symmetrically throughout the derivation. He derived nonlinear governing equations of motions for a Basic bicycle model with an active (leaning) rigid rider, and then linearized about the vertical equilibrium configuration. His equations of motion can be found as eq. XIV, eq. XV, and eq. XIII in his paper on pp. 321-323, but not all terms are defined. The equation are restated more clearly and explicitly in matrix form on p. 326. We also note that the figure defining some of his variables is at the end of the bound volume containing his paper.

[3] Comparisons to works by Whipple [1899], Carvallo [1901], Sommerfeld and Klein [1903], and Dohring [1955] were performed mainly by Dr. Jim Papadopoulos, whose results are summarized here. Some of his understanding and commentary on other comparisons are contained in other parts of this chapter.

It is most convenient to compare Whipple to Döhring [1955] since similar axis orientation is used. The equations on page 326 are in the form of the **3** x **3** matrix which operates on his variables ϕ, ϕ', τ, where λ is the derivative operator $\frac{d}{dt}$. There are a few evident typos: the first term of the second row should have λ^2 not λ_2; and the third column second row should have $W\gamma$, not $W'\gamma$) .

We found his notation to be more difficult to understand than most and therefore give some details about the comparison. In his notation,

$$\phi = \frac{\psi}{\sin\theta}$$

and

$$\phi' = \frac{\psi}{\sin\theta} - Q$$

where ψ is the lean of the rear frame and Q is the steer angle. (In our notation the lean of the rear frame is χ_r and ψ is the steer angle.) The last equation in his matrix defines τ as a function of ϕ and ϕ' and allows one to eliminate τ from the first two equations of the matrix. Doing so, one finds the first equation is in complete agreement with **our** lean equation when ϕ and ϕ' are written in terms of ψ and Q. The second equation of the matrix, when it is corrected and then multiplied by $\frac{\mu\mu'}{b\cos\theta}$, we find agrees completely with Döhring's [1955] equation (31). **As** is explained below, Döhring's equation **(31)** is a linear combination of **our** lean and steer equation and thus Whipple's linearized equations are in complete agreement with those presented in Chapter 111. His work, which is as sophisticated as almost any later investigation, was evidently done for his degree from Trinity College, Cambridge University.

Overall, the definitions of Whipple's variables are difficult to decipher and make his paper difficult to read, but his equations appear to be rigorously derived and are fully general when compared to those given in Chapter 111. Whipple is critical of McGaw's [1898] study of tricycles, and Bourlet's [pre-1896] study of bicycles, neither of which have we read.

Carvallo, 1901

Carvallo [1901] wrote 300 generally lucid pages on the stability of monocycles (rider inside a single wheel) and bicycles. Only the second part of the monograph, which won a prestigious prize, concerns us. In it he modifies Lagrange dynamics to deal with rolling hoops and bicycles (we were not able to tell if his method is a different way of dealing with nonholonomic constraints). We **are** concerned primarily with section V on no-hands stability. The equations where each term **was** derived are laid out on pp. 100-101, and restated in condensed form on p. 103. The equations are exactly analogous to ours, one for lean and one for steer.

Although we could not find where Carvallo said this, it appears that his bicycle has two identical heavy wheels, the rider and frame are considered a single unit, and the mass of the front assembly is at the center of the front wheel and its inertia properties are those of the wheel. (This is not an unreasonable idealization if the handlebars are not heavy and are positioned ***on*** the steering axis as was common in designs of that day. Technically, for **such** a design the mass of the handlebars and straight part of fork can then be considered part of the rear frame.)