PERSONAL   Code: SYN/N/2

LIST OF THESES SUBMITTED BY J. L. SYNGE FOR THE EXAMINATION FOR FELLOWSHIP IN APPLIED MATHEMATICS AT TRINITY COLLEGE, DUBLIN, 1925.

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Published Papers:

I. Principal Directions in a Riemannian Space.
   Principal Directions in the Einstein Solar Field.

   Proceedings of the National Academy of Sciences: Vol. [illegible], No. [illegible], pp. 19[illegible]-20[illegible], July 1922.

II. Elementary Notions of Space and Time.

   Journal of the Royal Astronomical Society of Canada: Vol. XVII, No. 6, pp. 2[illegible]-2[illegible]4, July-August 1923.

III. Parallel Propagation of a Vector around an infinitesimal Circuit in an affine-connected Manifold.

   Annals of Mathematics: Second Series, Vol. [illegible], No. 2, pp. 1[illegible]-1[illegible], December 1923.

IV. The Influence of the Earth's Rotation on a Top.

   Philosophical Magazine: Vol. XLVII, pp. [illegible], March 1924.

V. Principal Directions in an affine-connected Manifold of two Dimensions.

   Proceedings of the National Academy of Sciences: Vol. 1[illegible], No. 4, pp. 1[illegible]-1[illegible], April 1924.

Unpublished Papers:

VI. Applications of the Absolute Differential Calculus to the Theory of Elasticity.

   This paper will appear shortly in the Proceedings of the London Mathematical Society. 1925

VII. Improved Design for Automobile Steering Gear.

   This paper will appear shortly in "The Automobile Engineer".

[redacted]

IX. Normals and Curvatures of a Curve in the Riemannian Manifold

X. The First and Second Variations of the Length-Integral in Riemannian Space.

REMARKS

Of the ten theses submitted Nos. I, II, IV, VI, VII and VIII obviously deal with branches of Applied Mathematics. With regard to the other theses some remarks might not be out of place.

No. III: The subject matter of this paper is of some importance, because the method of parallel propagation is sometimes used to establish the tensorial character of the curvature tensor which is fundamental in the General Theory of Relativity. As remarked on p.182 the criticism of Weyl's method must be modified: his method is fundamentally sound, but in the form presented fails to carry conviction to the present writer.

No. V: This paper is directly connected with the world-geometry of Weyl. A mode of approach to this geometry conceptually clearer than that of Weyl will be found in a paper by Veblen and Thomas (Transactions of the American Mathematical Society, Vol.25, No.4, pp.551-608, October 1923), where reference may be found to a suggestion made by the present writer.

Nos. IX and X: With reference to these papers, as indeed also to those considered above, the remark may be made that any theorem in the geometry of an N-dimensional manifold may be expected to have a bearing on Applied Mathematics. The General Problem of Dynamics (Darboux: Lecons sur la Theorie Generale des Surfaces, Pt.II, p.480) is not essentially distinct from geometrical problems in such a manifold, and the present writer hopes later to consider further developments in the "Geometry of Dynamics", especially in connection with dynamical stability. The theorems in No.X concerning the "length-integral" can, of course, be translated directly into theorems concerning the "integral of action".

Thesis No. VIII

ON THE STABILITY OF THE BICYCLE.

John L. Synge
Feb. 7/1925

Copy No. 2

ON THE STABILITY OF THE BICYCLE.

By J. L. Synge

We can hardly hope to explain on simple mechanical grounds the stability acquired in walking, running and skating. Yet in view of the numerous phenomena displaying gyroscopic stability, the idea might be entertained that the accomplishment of riding a bicycle with the hands off the handlebars could be acquired by mere act of confidence and without subconscious (or conscious) muscular reactions resisting the disturbances; in fact, that the steady motion of a bicycle with a rigid rider rigidly attached to the frame might be one of dynamical stability. Owing to the non-holonomic character of this system, the full discussion (even after obvious simplifications) is of very great complexity; consequently we shall only discuss here certain idealised systems which, while not approximating closely to the actual configuration, may yet throw some light on the probable answer to the question raised. The mass of the rider, frame, forks, etc., we shall suppose concentrated into two equal masses, one situated at the centre of the back wheel, the other at the centre of the front wheel. When this is done, it is possible to consider the system as having no masses except in the wheels. It will be observed that the system so idealised is probably more stable than the actual system.

Specification. The system consists of four rigid bodies, viz.,

the back wheel (W),
the frame (F),
the forks (F'),
the front wheel (W').

W and W' are identical solids of revolution, each having a plane of symmetry perpendicular to the axis of revolution. F and F' are of zero mass. F has in common with W two points on the axis of revolution of the latter. There is a similar constraint between F' and W'. F and F' have two points in common, situated on the line of intersection of the planes of symmetry of W and W'. There are no frictional reactions at the joints.

The system is situated with the planes of symmetry of W and W' vertical and coincident, W and W' resting on a fixed, perfectly rough, horizontal plane (the ground). F and F' have a uniform motion of translation in a horizontal direction which lies in the common plane of symmetry of W and W'. This is evidently a state of steady motion.

The system receives a small impulsive disturbance which does not cause W or W' to leave the ground. The problem before us is the investigation of the stability of the resulting motion.

Let CXYZ be axes fixed in direction, C being the centre of W, CX horizontal in the direction of steady motion, and CZ directed vertically upwards.

Let Cxyz be moving axes, Cy being along the axis of revolution of W, and Cx being horizontal. In the steady motion these axes are coincident with CXYZ.

Let C'X'Y'Z' , C'x'y'z' be similarly defined by accenting the letters.

Let m = mass of W,
A, B, A = moments of inertia of W about Cxyz,
r = radius of section of W by plane of symmetry,
ρ = radius of curvature of the section of W by the plane $x = 0$, at the point $y = 0, z = -r$,
n = angular velocity of W in steady motion.

Let N and N' be the feet of the perpendiculars let fall from C and C' respectively on the common line of W and W'. Let

$p = CN, \quad p' = C'N',$
$h = N'N,$

$\alpha = N\hat{C}X = N'\hat{C}'X'$, in steady motion,
ℓ = CC', in steady motion,

$\psi = N\hat{C}x - \alpha,$

$\theta = \pi/2 - y\hat{C}Z,$

$\phi = x\hat{C}X,$
$\chi = \phi' - \phi,$
u, v, w, = velocities of C in the directions Cxyz,
$\omega_1, \omega_2, \omega_3$ = angular velocities of W about instantaneous positions of Cxyz,
$\theta_1, \theta_2, \theta_3$ = angular velocities of Cxyz about their instantaneous positions,
$\omega = \omega_2 - n.$

Similar definitions are obtained by accenting the letters.

We shall assume that $\theta, \theta', \dot{\phi}, \dot{\phi}', \psi, \psi', \chi, \omega, \omega'$ are small; we shall not however make use of the smallness of ϕ or ϕ'. The motion will be termed stable if the former nine quantities remain permanently small.

We easily deduce the following equations for the angular velocities in terms of the coordinates:-

$$\theta_1 = \dot{\theta}\ , \quad \theta_2 = 0, \quad \theta_3 = \dot{\phi}\ ; \quad \omega_1 = \dot{\theta}\ , \quad \omega_2 = n + \omega\ , \quad \omega_3 = \dot{\phi}\ .$$

The coordinates (x_0, y_0, z_0) of the point of contact of W with the ground are found to be approximately

$$x_0 = 0, \quad y_0 = -\rho\theta\ , \quad z_0 = -r,$$

and therefore the conditions of no slipping, viz.,

$$u - y_0\omega_3 + z_0\omega_2 = 0,$$
$$v - z_0\omega_1 + x_0\omega_3 = 0,$$
$$w - x_0\omega_2 + y_0\omega_1 = 0,$$

give

$$u = r(n + \omega), \quad v = -r\dot{\theta}, \quad w = 0.$$

The following scheme of direction cosines connects the directions of CXYZ, Cxyz and C'x'y'z':-

	X	Y	Z	x'	y'	z'
x	$\cos\phi$	$\sin\phi$	0	1	$-\chi$	0
y	$-\sin\phi$	$\cos\phi$	θ	χ	1	$\theta - \theta'$
z	$\theta\sin\phi$	$-\theta\cos\phi$	1	0	$-(\theta - \theta')$	1

We can find directly the velocities of N in the directions Cxyz and also in the directions C'x'y'z'; if we resolve the latter in the directions Cxyz and identify with the velocities already found directly, we obtain

$$(1)\quad \begin{cases} p\sin\alpha(\dot{\psi} - \dot{\psi}') = r(\omega - \omega'), \\ (r + p\sin\alpha)(\dot{\theta} - \dot{\theta}') + p\cos\alpha.\dot{\chi} + nr\chi = l\dot{\phi}', \\ p\cos\alpha(\dot{\psi} - \dot{\psi}') = -l\dot{\psi}' \end{cases}$$

Similarly, determining the direction cosines of N'N referred to Cxyz directly, and then indirectly by means of C'x'y'z', we obtain

$$\begin{cases} \sin(\alpha + \psi') = \sin(\alpha + \psi), \\ -\chi\sin\alpha + (\theta - \theta')\cos\alpha = 0, \\ \cos(\alpha + \psi') = \cos(\alpha + \psi). \end{cases}$$

These equations, with (1), comprise five independent relations, which may be written

$$(2)\quad \begin{cases} \dot{\psi} = 0, \quad \dot{\psi}' = 0, \quad \omega = \omega', \\ \chi\sin\alpha = (\theta - \theta')\cos\alpha, \\ (p + r\sin\alpha)\dot{\chi} + nr\cos\alpha.\chi = l\cos\alpha.\,\dot{\phi}'. \end{cases}$$

The system has eight coordinates with five relations between them, and therefore three degrees of freedom.

The force system on W is composed of the following elements:-

(i) The reaction of the ground, which may be resolved into forces (R_1, R_2, R_3) acting at (x_0, y_0, z_0) in the directions Cxyz. They are equivalent to forces (R_1, R_2, R_3) acting along Cxyz and couples $(-\rho\theta R_3 + rR_2, -rR_1, \rho\theta R_1)$ about Cxyz.

(ii) The reactions of F, which are equivalent to forces (F_1, F_2, F_3) acting along Cxyz and couples $(L_1, 0, L_3)$ about Cxyz.

(iii) The force of gravity, which may be resolved into forces $(0, -\theta mg, -mg)$ acting along Cxyz.

Hence the total force system on W consists of

$$\text{forces along Cxyz:} \begin{cases} R_1 + F_1, \\ R_2 + F_2 - \theta mg, \\ R_3 + F_3 - mg; \end{cases}$$

$$\text{couples about Cxyz:} \begin{cases} rR_2 - \rho\theta R_3 + L_1, \\ -rR_1, \\ \rho\theta R_1 + L_3. \end{cases}$$

The force system on W' is obtained by accenting the letters.

There are certain relations connecting the force-couple systems

$(F_1, F_2, F_3; L_1, 0, L_3)$ referred to Cxyz, and

$(F'_1, F'_2, F'_3; L'_1, 0, L'_3)$ referred to C'x'y'z',

for, since F and F' are of zero mass, the above systems together form a system in statical equilibrium. Thus, observing that all these forces and couples are small, we obtain the relations

$$(3) \qquad \begin{cases} F_1 + F'_1 = 0, & L_1 + L'_1 = 0, \\ F_2 + F'_2 = 0, & F'_3 = 0, \\ F_3 + F'_3 = 0, & L_3 + L'_3 + \ell F'_2 = 0. \end{cases}$$

To these we add another relation deduced from the fact that the reactions between F and F' have no moment about N'N, viz.,

$$(4) \qquad pF_2 + L_1 \sin\alpha - L_3 \cos\alpha = 0.$$

The dynamical equations of W,

$$\left\{\begin{aligned}
&m(\dot u - v\theta_3 + w\theta_2) = \text{component of force in direction } Cx,\\
&m(\dot v - w\theta_1 + u\theta_3) = \text{component of force in direction } Cy,\\
&m(\dot w - u\theta_2 + v\theta_1) = \text{component of force in direction } Cz,\\
&A\dot\omega_1 - B\omega_2\theta_3 + A\omega_3\theta_2 = \text{moment about } Cx,\\
&B\dot\omega_2 - A\omega_3\theta_1 + A\omega_1\theta_3 = \text{moment about } Cy,\\
&A\dot\omega_3 - A\omega_1\theta_2 + B\omega_2\theta_1 = \text{moment about } Cz,
\end{aligned}\right.$$

become, on substitution of coordinates and forces,

$$\left\{\begin{aligned}
&mr\dot\omega &&= R_1 + F_1,\\
&m(-r\ddot\theta + nr\dot\phi) &&= R_2 + F_2 - mg\theta,\\
&0 &&= R_3 + F_3 - mg,\\
&A\ddot\theta - Bn\dot\phi &&= rR_2 - \rho\theta R_3 + L_1,\\
&B\dot\omega &&= -rR_1,\\
&A\ddot\phi + Bn\dot\theta &&= \rho\theta R_1 + L_3.
\end{aligned}\right.$$

Eliminating R_1, R_2 and R_3 we obtain

$$(5)\qquad \left\{\begin{aligned}
&(B + mr^2)\dot\omega = rF_1,\\
&(A + mr^2)\ddot\theta + mg(\rho - r)\theta - n(B + mr^2)\dot\phi = -rF_2 + L_1,\\
&A\ddot\phi + Bn\dot\theta = L_3,
\end{aligned}\right.$$

to which we add the corresponding equations for W',

$$(6)\qquad \left\{\begin{aligned}
&(B + mr^2)\dot\omega' = rF_1',\\
&(A + mr^2)\ddot\theta' + mg(\rho - r)\theta' - n(B + mr^2)\dot\phi' = -rF_2' + L_1',\\
&A\ddot\phi' + Bn\dot\theta' = L_3'.
\end{aligned}\right.$$

Eliminating the ten forces and couples from the thirteen equations, (3),(4),(5) and (6), we obtain the three equations

$$(7)\qquad \left\{\begin{aligned}
&\dot\omega + \dot\omega' = 0,\\
&(A + mr^2)(\ddot\theta + \ddot\theta') + mg(\rho - r)(\theta + \theta')\\
&\qquad - n(B + mr^2)(\dot\phi + \dot\phi') = 0,\\
&(\rho + r\sin\alpha)\left[A(\ddot\phi + \ddot\phi') + Bn(\dot\theta + \dot\theta')\right]\\
&\quad + \ell\sin\alpha\left[(A + mr^2)\ddot\theta + mg(\rho - r)\theta - n(B + mr^2)\dot\phi\right]\\
&\qquad - \ell\cos\alpha(A\ddot\phi + Bn\dot\theta) = 0.
\end{aligned}\right.$$

Writing

$$\Phi = \phi' + \phi,$$
$$\Theta = \theta' + \theta,$$
$$\eta = \theta' - \theta,$$

we see from equations (2) and (7) that the complete solution of the problem lies in the equations

$$(8)\left\{\begin{aligned}
&\psi = 0\ ,\\
&\psi' = 0\ ,\\
&\omega = \omega',\\
&\chi \sin\alpha = -\eta\cos\alpha,\\
&P\dot{\chi} + 2nr\cos\alpha\,.\chi = \ell\cos\alpha\,.\dot{\Phi}\ ,\\
&(A+mr^2)\ddot{\Theta} + mg(\rho - r)\Theta - n(B+mr^2)\dot{\Phi} = 0\ ,\\
&AP\ddot{\Phi} + BPn\dot{\Theta} + \ell\cos\alpha\,.A\ddot{\chi} + \ell\sin\alpha\,(B+mr^2)n\dot{\chi}\\
&\quad - \ell\sin\alpha\,(A+mr^2)\ddot{\eta} + \ell\cos\alpha\,.Bn\dot{\eta} - \ell\sin\alpha.mg(\rho - r)\eta = 0,\\
&\dot{\omega} + \dot{\omega}' = 0\ ,
\end{aligned}\right.$$

where $P = p + p' + 2r\sin\alpha$. We have at once the solutions for four of the variables, viz.,

$$\psi = \psi' = 0\ , \qquad \omega = \omega' = \text{constant}.$$

There are four equations left for Φ, Θ, η, χ . Eliminating the first three variables, we obtain

$$(9)\qquad K_4\ddddot{\chi} + K_3\dddot{\chi} + K_2\ddot{\chi} + K_1\dot{\chi} + K_0\chi = 0\ ,$$

where

$$K_4 = (A+mr^2)\left[A(P^2+\ell^2) + \ell^2 mr^2\sin^2\alpha\right]\ ,$$
$$K_3 = nr\cos\alpha(A+mr^2)(2AP + \ell^2 mr\sin\alpha)\ ,$$
$$K_2 = NP^2 + \ell^2 mg(\rho - r)\left[A(1+\sin^2\alpha) + 2mr^2\sin^2\alpha\right]\ ,$$
$$K_1 = nr\cos\alpha\left[2NP + \ell^2 mr\sin\alpha\,.mg(\rho - r)\right]\ ,$$
$$K_0 = \ell^2\sin^2\alpha\left[mg(\rho - r)\right]^2,$$
$$N = Amg(\rho - r) + n^2B(B+mr^2)\ .$$

It is easily seen that the criterion for the stability of the system is the stability of the roots of (9). Now the conditions that the roots of (9) should be stable, or, what is the same thing, that the roots of

$$K_4 z^4 + K_3 z^3 + K_2 z^2 + K_1 z + K_0 = 0$$

should have their real parts negative or zero, are that

$$K_0,\ K_1,\ K_2,\ K_3,\ K_4,\ K_1K_2K_3 - K_0K_3^2 - K_1^2K_4$$

should all have the same sign (See Routh: Advanced Rigid Dynamics (1905), p.222). Since K_0 and K_4 are obviously positive, the stability conditions become

$$K_1 > 0,\quad K_2 > 0,\quad K_3 > 0,\quad K_1K_2K_3 - K_0K_3^2 - K_1^2K_4 > 0.$$

We shall assume

$$0 < \alpha < \pi/2, \qquad P = p + p' + 2r \sin\alpha > 0.$$

If we put

$$a_4 = (A + mr^2)\left[A(P^2 + \ell^2) + \ell^2 mr^2 \sin^2\alpha\right],$$

$$a_3 = (A + mr^2)(2AP + \ell^2 mr \sin\alpha),$$

$$P^2 a_2 = -\ell^2 mg(\rho - r)\left[A(1 + \sin^2\alpha) + 2mr^2 \sin^2\alpha\right],$$

$$2Pa_1 = -\ell^2 mr \sin\alpha . mg(\rho - r),$$

$$a_0 = \ell^2 \sin^2\alpha\left[mg(\rho - r)\right]^2,$$

we shall have

$$K_4 = a_4,$$

$$K_3 = nr \cos\alpha . a_3,$$

$$K_2 = P^2(N - a_2),$$

$$K_1 = nr \cos\alpha . 2P(N - a_1),$$

$$K_0 = a_0,$$

and the conditions for stability may be written

(10) $$n > 0,\quad N > a_1,\quad N > a_2,\quad f(N) > 0,$$

where

$$f(N) = 2P^3 a_3 (N - a_1)(N - a_2) - a_0 a_3^2 - 4P^2 a_4 (N - a_1)^2.$$

The coefficient of N^2 in $f(N)$ is $2P^2(Pa_3 - 2a_4)$, or

$$2P^2\ell^2(A + mr^2)\left[mr \sin\alpha\,(p + p') - 2A\right].$$

Putting $N - a_1 = \nu$, the equation $f(N) = 0$ becomes

$$2P^2(Pa_3 - 2a_4)\nu^2 + 2P^3 a_3(a_1 - a_2)\nu - a_0 a_3^2 = 0.$$

Observing that

$$2P^2(a_1 - a_2) = \ell \sin\alpha . mg(\rho - r)\left[-\frac{(Pa_3 - 2a_4)}{\ell \sin\alpha\ (A + mr^2)} + 2\ell \sin\alpha (A+mr^2)\right],$$

we find the roots of the above quadratic to be

$$\nu_1 = \frac{a_3 mg(\rho - r)}{2P(A + mr^2)}$$

$$\nu_2 = -\frac{a_3 \sin^2\alpha\ .mg(\rho - r)}{P[mr\ \sin\alpha(p + p') - 2A]}$$

Hence, if N_1, N_2 are the roots of $f(N) = 0$,

$$N_1 = a_1 + \nu_1 = \quad Amg(\rho - r) \ ,$$

$$N_2 = a_1 + \nu_2 = \quad Amg(\rho - r) - \frac{mg(\rho - r)a_3}{2P(A + mr^2)}\left[\frac{2\ \sin^2\alpha\ (A + mr^2)}{mr\ \sin\alpha(p + p') - 2A} + 1\right].$$

<u>Case I:</u> $\qquad mr\ \sin\alpha\ (p + p') - 2A > 0$

We have

$$f(a_1) = -a_0 a_3^2 < 0,$$

$$f(a_2) = -a_0 a_3^2 - 4P^2 a_4(a_1 - a_2)^2 < 0.$$

Therefore, since the coefficient of N^2 in $f(N)$ is positive, a_1 and a_2 lie between N_1 and N_2. Hence the conditions (10) for stability are satisfied if, and only if, n is positive and N exceeds both N_1 and N_2. We deduce the following two results:-

(A) If

$$0 < \alpha < \pi/2,$$

$$p + p' + 2r\ \sin\alpha > 0,$$

$$mr\ \sin\alpha(p + p') - 2A > 0,$$

$$\rho > r,$$

then the necessary and sufficient condition for stability is

$$n > 0.$$

$$
\text{(B)}\left\{
\begin{array}{l}
\text{If} \quad 0 < \alpha < \pi/2, \\
\quad p + p' + 2r\sin\alpha > 0, \\
\quad mr\sin\alpha(p + p') - 2A > 0, \\
\quad \rho < r, \\
\text{then the necessary and sufficient conditions for stability are} \\
\quad n > 0, \\
n^2 B(B + mr^2) > mg(r-\rho)\left[A + \dfrac{mr\ell^2\sin\alpha}{2(p+p'+2r\sin\alpha)}\right]\left[1 + \dfrac{2\sin^2\alpha(A+mr^2)}{mr\sin\alpha(p+p')-2A}\right]
\end{array}
\right.
$$

Case II: $\quad mr\sin\alpha\,(p + p') - 2A < 0$

In this case we observe that $(a_1 - a_2)$, ν_1, ν_2 all carry the sign of $(\rho - r)$; also the coefficient of N^2 in $f(N)$ is negative. The condition

$$f(N) > 0$$

is satisfied if, and only if, N lies between N_1 and N_2. Now if $\rho < r$, then ν_1 and ν_2 are negative and

$$N_1 < a_1, \qquad N_2 < a_1.$$

Therefore the conditions of stability

$$f(N) > 0, \qquad N > a_1,$$

are incompatible, and we have the result:-

$$
\text{(C)}\left\{
\begin{array}{l}
\text{If} \quad 0 < \alpha < \pi/2, \\
p + p' + 2r\sin\alpha > 0, \\
mr\sin\alpha\,(p + p') - 2A < 0, \\
\rho < r, \\
\text{then the steady motion is unstable for all values of } n.
\end{array}
\right.
$$

If, on the other hand, $\rho > r$, then $(a_1 - a_2)$, ν_1, ν_2 are all positive and

$$N_1 > a_1 > a_2, \qquad N_2 > a_1 > a_2.$$

Hence the conditions of stability reduce to two, viz., $n > 0$ and that N should lie between N_1 and N_2. Thus we have the result:-

(D) If

$$0 < \alpha < \pi/2,$$

$$p + p' + 2r \sin\alpha > 0,$$

$$mr \sin\alpha\,(p + p') - 2A < 0,$$

$$\rho > r,$$

then the necessary and sufficient conditions for stability are

$$n > 0,$$

$$n^2 B(B + mr^2) < -mg(\rho - r)\left[A + \frac{mr\ell^2 \sin\alpha}{2(p+p'+2r\sin\alpha)}\right]\left[1 + \frac{2\sin^2\alpha\,(A + mr^2)}{mr\sin\alpha(p+p') - 2A}\right]$$

The results lettered (A),(B),(C),(D) comprise the results of our investigation. Result (D) is remarkable in that the motion may be stable at low but not at high speeds. However, in connection with the ordinary bicycle (for which $\rho < r$), we are concerned with (B) and (C). Here we observe theoretical support for the well-known fact that a long wheel base is conducive to stability. In fact we may state for our idealised system that if the structure of the wheels is given and the position of the head relative to the front wheel, it is only necessary to increase p (and consequently the wheel base) sufficiently in order to make the machine pass from the class (C) to the class (B), i.e. from the class of absolute instability to the class of stability at high speeds. We also observe that, generally speaking, a fairly large angle of rake (α) is conducive to stability in just the same way.

To give our results a more practical colour, we shall now assume that each wheel consists of a rim of mass μ and a concentrated mass $(m - \mu)$ at the centre, μ being small in comparison with m and consequently neglected in comparison with it. We have then

$$A = \tfrac{1}{2}\mu r^2, \qquad B = \mu r^2.$$

Further, let us put $\rho = 0$, and suppose that in steady motion the line of the head passes through the point of contact of the front wheel with the ground - a state of affairs approximately existent in most machines. We deduce

$$p' + r\sin\alpha = 0,$$

and we have in general

$$p - p' = \ell \cos\alpha .$$

We have then approximately

$$mr \sin\alpha(p+p') - 2A = mr \sin\alpha(\ell\cos\alpha - 2r \sin\alpha).$$

Since $\ell > 2r$ and $\alpha < \pi/4$ in practical cases, this quantity is positive and consequently result (B) is applicable. Hence we have as the condition for stability

$$n > \frac{\ell}{r}\sqrt{\frac{mg \sin\alpha}{2\mu(\ell\cos\alpha - 2r \sin\alpha)}}$$

From this result - if it be not too rash to make inference respecting the practical problem from the idealised configuration - we may conclude that at sufficiently high speeds the gyroscopic action of the wheels stabilises the system without action on the part of the rider.

We shall conclude with the specification of systems to illustrate the four results (A), (B), (C), and (D).

(i) Let $\alpha = \pi/6$, $\ell = 6r$, $p' + r \sin\alpha = 0$, and let W be a solid homogeneous ellipsoid of revolution, the semi-axis of revolution being $2r$. Result (A) is applicable, and the stability condition is $n > 0$.

(ii) Let $\alpha = \pi/6$, $\ell = 3r$, $p' + r \sin\alpha = 0$, and let W be a thin uniform disc. Result (B) is applicable, and the stability condition is

$$n > \sqrt{\frac{g}{2r}(4 + 3\sqrt{3})}$$

(iii) Let $\alpha = \pi/6$, $\ell = 3r$, $p' + r \sin\alpha = 0$, and let W be a thin ring. Result (C) is applicable, and the motion is unstable at all speeds.

(iv) Let $\alpha = \pi/6$, $\ell = 3r$, $p' + r \sin\alpha = 0$, and let W be as in (i). Result (D) is applicable, and the motion is unstable at all speeds.

(v) Let $\alpha = \pi/6$, $\ell = 5r$, $p' + r \sin\alpha = 0$, and let W be as in (i). Result (D) is applicable, and the stability condition is

$$0 < n < \sqrt{\frac{195g}{28r}(2 + \sqrt{3})}$$

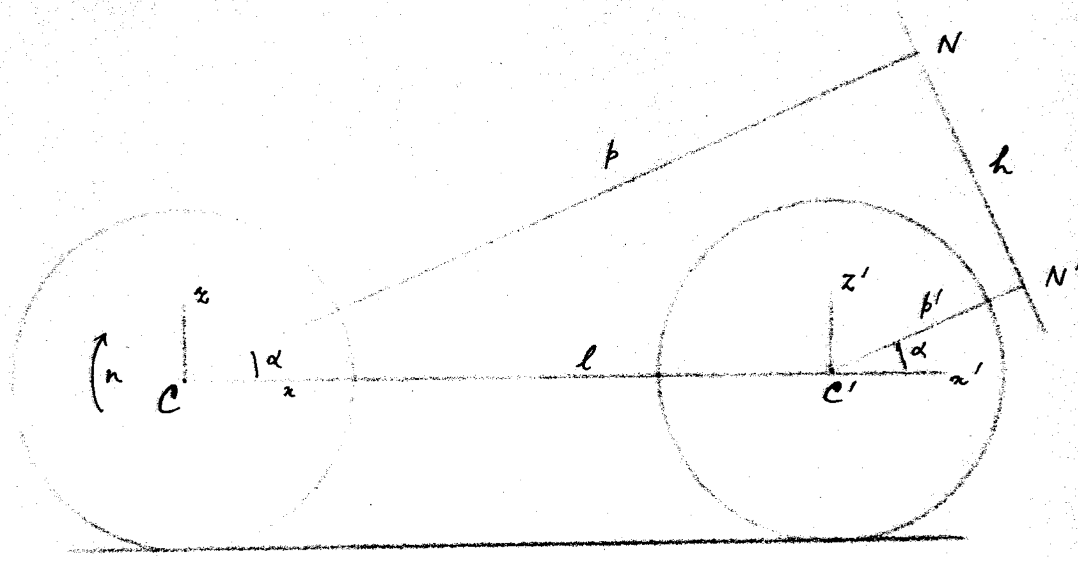

Fig. 1 Steady Motion.

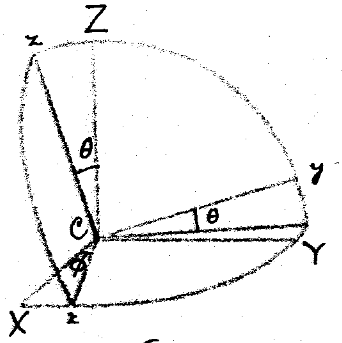

Fig. 2