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LINEARIZED EQUATIONS FOR AN EXTENDED BICYCLE MODEL

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Abstract. *The linearized equations of motion for a bicycle of the usual construction travelling straight ahead on a level surface have been the subject of several previous studies. In the simplest models, the pure-rolling conditions of the knife-edge wheels are introduced as non-holonomic constraints and the rider is assumed to be rigidly attached to the rear frame. There are two degrees of freedom for the lateral motion, the lean angle of the rear frame and the steering angle. In the present paper, the model is extended in several ways, while the simplicity of having only two degrees of freedom is retained. The extensions of the model comprise the shape of the tires, which are allowed to have a finite transverse radius of curvature, the effect of a pneumatic trail and a damping term due to normal spin at the tire contact patch, the gradient of the road, the inclusion of driving and braking torques at the wheels and the aerodynamic drag at the rear frame.*

Owing to the gradient, the yaw angle of the rear frame is no longer a cyclic coordinate and the kinematic differential equation for its evolution needs to be included. A further consequence is that the stiffness matrix is no longer symmetric, even for zero speed and acceleration. The way of decelerating has a marked influence on the stability characteristics: braking at the rear wheel, braking at the front wheel, aerodynamic drag and riding up an incline influence the lateral dynamics in different ways. The acceleration makes the coefficients of the linearized system time-varying.

A comparison of the derived equations and the results obtained by a multibody dynamic program is made, which shows a complete agreement. The equations can be used for several purposes: firstly, they provide a non-trivial example of a non-holonomic system that can be used to illustrate some of the characteristic properties of systems of this kind; secondly, they can be used as a test problem for the verification of multibody dynamic codes; thirdly, the simple model already yields valuable insight in the effects of several system parameters on the dynamics of a bicycle.

1 INTRODUCTION

In the literature, the linearized equations for the motion of a bicycle have been the subject of several studies. The classic article by Whipple [1] considered the motion of a bicycle on a level road where all bodies are considered rigid and the contact between the wheels and the road is described by holonomic constraints in the normal direction and non-holonomic constraints that impose a condition of pure rolling in the tangential directions. The wheels are further assumed to have knife-edge rims. An overview of further contributions and a verification of the linearized equations can be found in [2].

Models for motorcycles can have a similar structure [3], or they may include tire force models [4]. Further developments of models have been made, in which degrees of freedom were added and more accurate tire force models were used. An extensive analytic model was presented by Koenen [5], which also considered stationary cornering. In the last decade, the analysis has shifted from analytically derived models to numeric or symbolic models generated with multibody dynamic programs [6, 7].

Here, the attention is focused on the dynamics of a bicycle in the low-speed range, in which structural flexibility and tire slips are of minor importance [8]. Extensions are made to the original bicycle model which do not increase the number of degrees of freedom, so a more accurate model can be obtained from which conclusions can still easily be extracted. A first extension concerns the finite transverse radius of curvature of the crown of the tire. Without loss of generality, the outer shape of the wheel is considered to be toroidal. Toroidal wheels have already been considered in [9] in a kinematical analysis with ideal constraints and in [6] as part of a tire force model. A second extension considers the accelerated bicycle. The acceleration can be caused by a road gradient, by moments at the hubs of the rear and front wheel and by aerodynamic drag.

The paper is organized as follows. First, the bicycle model is described. Then the linearized equations of motion for lateral perturbations of a nominal longitudinal motion are derived. The results are compared with numerical results obtained from a multibody dynamic program. Finally, some interesting observations are made.

2 BICYCLE MODEL

Figure 1 shows the construction, the main dimensions and the coordinates of the bicycle. The model consists of four rigid bodies that are connected by revolute joints. The rear frame is connected to the front fork at the steering head, the rear wheel is connected to the rear frame at the rear wheel hub and the front wheel is connected to the front fork at the front wheel hub. The rider is assumed to be rigidly connected to the rear frame. The bicycle moves on a rigid plane road surface. The global coordinate system has its x- and y-axis in the road surface and its z-axis pointing downwards. In the reference configuration of the bicycle, all four bodies have their body-fixed coordinate axes aligned with the corresponding axes of the global coordinate system, the wheels just touch the road surface and the origin of the rear frame is in the origin of the global coordinate system. The wheels have a toroidal tire tread surface with which they can be in contact with the road surface. The local y-axis of either wheel is along the axis of rotational symmetry and the local x- and z-axis are in the meridional plane of symmetry. The wheel radius in this plane is r_{r} or r_{f} for the rear or front wheel respectively and the respective transverse curvatures are ρ_{r} and ρ_{f}. The rear frame has its local x- and z-axis in its plane of symmetry and its y-axis perpendicular to this plane; the centre of the rear wheel hub has coordinates $(0, 0, -r_{\mathrm{r}})$. The front fork coordinate system has its origin on the steering axis at

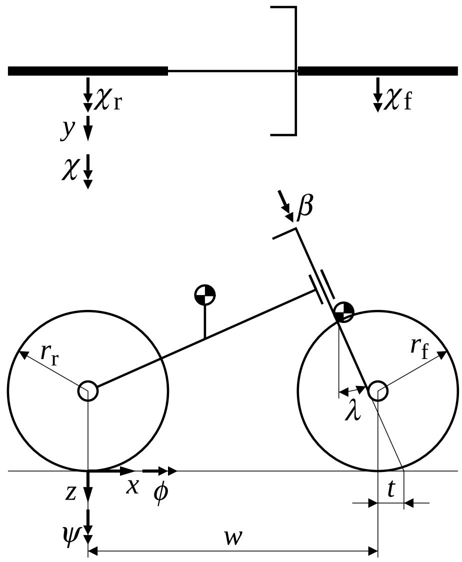

Figure 1: Bicycle model with coordinates and main dimensions.

the point $(w + t, 0, 0)$ in the coordinate system of the rear frame, which is in the road surface in the reference configuration; w is the wheelbase and t the trail, so the front wheel touches the road at the point $(w, 0, 0)$ in the reference configuration; the front wheel hub is at $(-t, 0, -r_{\mathrm{f}})$ in the front fork coordinate system. The steering axis makes an angle λ, the rake angle, with the z-axis, as is indicated in Figure 1.

An arbitrary configuration of the bicycle can be described by nine generalized coordinates. The position of the origin of the rear frame in the global coordinate system can be described by three Cartesian coordinates, x, y and z. Its orientation can be described by three angles. The angle between the global x-axis and the line of intersection of the symmetry plane of the rear frame with the road surface is the yaw angle ψ, where a positive angle corresponds to a positive rotation about the z-axis. The lean angle ϕ is the angle that the symmetry plane makes with the normal to the road surface, where a lean to the right yields a positive angle. The pitch angle χ is defined as the angle between the local x-axis and the line of intersection of the symmetry plane with the road surface. The other three coordinates are the relative angles of rotation at the steering head, β, and at the wheel hubs, χ_{r} and χ_{f}. Four of these coordinates are cyclic: the two coordinates x and y and the wheel rotation angles χ_{r} and χ_{f}. The yaw angle ψ is also a cyclic coordinate if the road surface is level.

The generalized coordinates are only independent if the bicycle floats freely in the air. It will be assumed, however, that the the two wheels remain in contact with the road surface, which imposes two holonomic constraints. The heave z and the pitch angle χ are considered to be dependent coordinates, which are functions of the lean angle ϕ and the steering angle β. Furthermore, no-slip conditions are imposed, which yield four non-holonomic constraints on the velocities. Ideally, the constraints impose zero longitudinal and lateral slips at the contact points. A small modification is made here for the lateral slip, in order to include the effects of the so-called pneumatic trail. For a non-ideal contact, the linearized relation between the lateral force and aligning moment and the lateral slip and normal spin rate of the wheel can be approximated by the relations

$$F_y = C_y(\dot\epsilon_y - t_\mathrm{p}\omega_\mathrm{n})/V_x, \qquad (1)$$
$$M_\mathrm{n} = C_y t_\mathrm{p}\dot\epsilon_y/V_x.$$

Here, $\dot\epsilon_y$ is the lateral slip velocity, ω_n is the normal spin rate, that is, the component of the angular velocity of the wheel in the direction normal to the road surface, V_x is the forward velocity of the wheel, F_y is the lateral tyre force, M_n is the aligning moment, C_y is the cornering stiffness and t_p is the pneumatic trail. For pure sideslip, the lateral force has a line of action that is a distance t_p behind the ideal point of contact. Hence, it seems to be appropriate to impose, for low speeds, a combined zero-slip condition,

$$\dot\epsilon'_y = \dot\epsilon_y + t_\mathrm{p}\omega_\mathrm{n} = 0. \qquad (2)$$

In general, the relation between the aligning moment, the lateral force and the normal spin rate is

$$M_\mathrm{n} = t_\mathrm{p}F_y + C_y t_\mathrm{p}^2\omega_\mathrm{n}/V_x. \qquad (3)$$

The first term is automatically accounted for by the modified constraint (2), and the second term can be added as a viscous damping term. The non-slip condition in the longitudinal direction does not need any modification. The independent velocities are the rear wheel rotation rate $\dot\chi_\mathrm{r}$, the lean rate, $\dot\phi$, and the steering rate, $\dot\beta$. Instead of the rear wheel rotation rate, the corresponding forward speed $v = -r_\mathrm{r}\dot\chi$ will often be used.

The road surface may be inclined with respect to the direction of the gravity field with an angle α. This means that the gravity field has a component $g_x = g\sin\alpha$ in the global x-direction and a component $g_z = g\cos\alpha$ in the global z-direction.

The bicycle may be accelerated or braked by a moment M_r between the rear frame and the rear wheel and a moment M_f between the front fork and the front wheel. Furthermore, a pure drag force F_d is assumed that acts at the pressure point of the rear frame and has a direction opposite to the absolute velocity of this pressure point, $\dot{\mathbf{x}}_\mathrm{p}$, with a magnitude

$$F_\mathrm{d} = \tfrac{1}{2}\rho_\mathrm{air}C_\mathrm{d}Av_\mathrm{p}^2. \qquad (4)$$

Here, ρ_air is the density of the surrounding air, C_d is the drag coefficient, A is the frontal area and v_p is the magnitude of $\dot{\mathbf{x}}_\mathrm{p}$.

The geometric data, the masses and moments of inertia of the four bodies and some additional parameters for an example bicycle are given in Table 1. The inclination angle of the road, the driving and braking moments and the forward velocity are variable.

2.1 Nominal solution

In the nominal solution, all out-of-plane motions vanish. This means that the coordinates y, ψ, ϕ and β are identically equal to zero. Also the dependent coordinates, the heave z and the pitch angle χ, are identically equal to zero, owing to the holonomic constraints. The only dynamic degree of freedom is the forward motion, described by χ_r, while the forward velocity and the rotation rate of the front wheel is directly coupled to this motion as $\dot x = v = -r_\mathrm{r}\dot\chi_\mathrm{r}$ and $\dot\chi_\mathrm{f} = \dot\chi_\mathrm{r}r_\mathrm{r}/r_\mathrm{f}$. Similarly for the virtual displacements, $\delta x = -r_\mathrm{r}\delta\chi_\mathrm{r}$ and $\delta\chi_\mathrm{f} = \delta\chi_\mathrm{r}r_\mathrm{r}/r_\mathrm{f}$.

For further reference, the total mass, m_T, and the coordinates of the centre of mass of the bicycle as a whole, $(x_\mathrm{T}, 0, z_\mathrm{T})$, are defined as

$$\begin{aligned} m_\mathrm{T} &= m_\mathrm{r} + m_\mathrm{f} + m_\mathrm{rf} + m_\mathrm{ff}, \\ x_\mathrm{T} &= (m_\mathrm{f}w + m_\mathrm{rf}x_\mathrm{rf} + m_\mathrm{ff}x_\mathrm{ff})/m_\mathrm{T}, \\ z_\mathrm{T} &= (-m_\mathrm{r}r_\mathrm{r} - m_\mathrm{f}r_\mathrm{f} + m_\mathrm{rf}z_\mathrm{rf} + m_\mathrm{ff}z_\mathrm{ff})/m_\mathrm{T}. \end{aligned} \qquad (5)$$

Symbol	Meaning	Value
w	wheelbase	1.02 m
t	trail	0.08 m
λ	steer inclination angle	0.1π rad
g	strength of gravity field	9.81 N/kg
r_r	rear wheel radius	0.3 m
ρ_r	rear wheel crown radius	0.02 m
m_r	rear wheel mass	2 kg
$I_{\mathrm{r}xx} = I_{\mathrm{r}zz}$, $I_{\mathrm{r}yy}$	rear wheel mass moments of inertia	$0.0603, 0.12$ kgm^2
t_pr	rear wheel pneumatic trail	0.018 m
$C_{y\mathrm{r}}$	rear wheel cornering stiffness	2500 N
r_f	front wheel radius	0.35 m
ρ_f	front wheel crown radius	0.015 m
m_f	front wheel mass	3 kg
$I_{\mathrm{f}xx} = I_{\mathrm{f}zz}$, $I_{\mathrm{f}yy}$	front wheel mass moments of inertia	$0.1405, 0.28$ kgm^2
t_pf	front wheel pneumatic trail	0.012 m
$C_{y\mathrm{f}}$	front wheel cornering stiffness	1500 N
x_rf, z_rf	position of rear frame centre of mass	0.3, -0.9 m
m_rf	rear frame mass	85 kg
$I_{\mathrm{rf}xx}$, $I_{\mathrm{rf}xz}$, $I_{\mathrm{rf}yy}$, $I_{\mathrm{rf}zz}$	rear frame moments of inertia	9.2, 2.4, 11, 2.8 kgm^2
x_ff, z_ff	position of front fork centre of mass	0.9, -0.7 m
m_ff	front fork mass	4 kg
$I_{\mathrm{ff}xx}$, $I_{\mathrm{ff}xz}$, $I_{\mathrm{ff}yy}$, $I_{\mathrm{ff}zz}$	front fork moments of inertia	0.05892, -0.00756, 0.06, 0.00708 kgm^2
ρ_air	density of the air	1.2 kg/m^3
$C_\mathrm{d}A$	aerodynamic frontal area	0.4 m^2
x_d, z_d	coordinates of the pressure point	0.4, -0.8 m

Table 1: Parameter values for the example bicycle.

The equation of motion can now easily be derived as

$$(m_\mathrm{T} + I_{\mathrm{r}yy}/r_\mathrm{r}^2 + I_{\mathrm{f}yy}/r_\mathrm{f}^2)\dot{v} = m_\mathrm{T} g_x + M_\mathrm{r}/r_\mathrm{r} + M_\mathrm{f}/r_\mathrm{f} - F_\mathrm{d}, \tag{6}$$

where $F_\mathrm{d} = \frac{1}{2}\rho_\mathrm{air} C_\mathrm{d} A v^2$. The expression between brackets on the left-hand side can be called an effective mass, which contains a contribution due to the moments of inertia of the wheels. The normal forces at the wheel contact points are easily found from the momentum and angular momentum balance of the bicycle as a whole as

$$\begin{aligned} F_{z\mathrm{r}} &= \frac{1}{w}\big[m_\mathrm{T} z_\mathrm{T}(g_x - \dot{v}) + m_\mathrm{T}(w - x_\mathrm{T})g_z - z_\mathrm{p} F_\mathrm{d} + (I_{\mathrm{r}yy}/r_\mathrm{r} + I_{\mathrm{f}yy}/r_\mathrm{f})\dot{v}\big], \\ F_{z\mathrm{f}} &= \frac{1}{w}\big[-m_\mathrm{T} z_\mathrm{T}(g_x - \dot{v}) + m_\mathrm{T} x_\mathrm{T} g_z + z_\mathrm{p} F_\mathrm{d} - (I_{\mathrm{r}yy}/r_\mathrm{r} + I_{\mathrm{f}yy}/r_\mathrm{f})\dot{v}\big]. \end{aligned} \tag{7}$$

The longitudinal forces follow from the angular momentum balance of the wheels as

$$\begin{aligned} F_{x\mathrm{r}} &= (I_{\mathrm{r}yy}/r_\mathrm{r}^2)\dot{v} - M_\mathrm{r}/r_\mathrm{r}, \\ F_{x\mathrm{f}} &= (I_{\mathrm{f}yy}/r_\mathrm{f}^2)\dot{v} - M_\mathrm{f}/r_\mathrm{f}. \end{aligned} \tag{8}$$

The normal forces might be needed for an estimation of the pneumatic trails.

3 LINEARIZED EQUATIONS OF MOTION

In this section the linearized equations for the bicycle for lateral perturbations of the nominal in-plane motion are derived. Owing to the symmetry of the system, there is no linear coupling between the lateral (out-of-plane) motion and the longitudinal (in-plane) motion. So the non-linear longitudinal equation for the nominal motion is retained and lateral perturbations about this time-varying motion are considered. Successively, the configuration, the velocities, the virtual displacements and the accelerations are analysed. Finally the linearized equations are derived with the principle of virtual work.

3.1 Configuration analysis

If linearized equations have to be derived, only a first-order analysis is needed. For the virtual displacements, however, to be discussed later on, these expressions appear in a linearized form and second-order terms yield virtual displacements that have coefficients containing terms that are linear in the coordinates. When these are multiplied with the large longitudinal forces, they contribute to the linearized equations. Because of the linear decoupling of the lateral and longitudinal motion, the heave and pitch angle do not have a linear dependence on the lateral coordinates. In the sequel, terms of higher than second order are dropped without further notice.

The rotation matrix for the rear frame can be expanded to second order as

$$\mathbf{R}_\mathrm{r} = \begin{pmatrix} 1-\frac{1}{2}\psi^2 & -\psi & \chi+\psi\phi \\ \psi & 1-\frac{1}{2}\phi^2-\frac{1}{2}\psi^2 & -\phi \\ -\chi & \phi & 1-\frac{1}{2}\phi^2 \end{pmatrix}. \tag{9}$$

The z-coordinate of the centre of the rear wheel and its inclination angle are

$$z_{0\mathrm{r}} = z - (1-\tfrac{1}{2}\phi^2)r_\mathrm{r}, \quad \gamma_\mathrm{r} = \phi. \tag{10}$$

On the other hand, the z-coordinate that follows from the holonomic constraint is

$$z_{0\mathrm{r}} = -\rho_\mathrm{r} - (r_\mathrm{r}-\rho_\mathrm{r})\cos\gamma_\mathrm{r} \approx -(1-\tfrac{1}{2}\phi^2)r_\mathrm{r} - \tfrac{1}{2}\phi^2\rho_\mathrm{r}, \tag{11}$$

from which it can be concluded that

$$z = -\tfrac{1}{2}\rho_\mathrm{r}\phi^2. \tag{12}$$

Note that this constraint does not involve the pitch angle.

The rotation matrix of the front fork with respect to the rear frame depends on the rake angle λ, which is not small. The expanded relative rotation matrix is

$$\mathbf{R}_{\mathrm{f/r}} = \begin{pmatrix} 1-\frac{1}{2}\beta^2\cos^2\lambda & -\beta\cos\lambda & \frac{1}{2}\beta^2\sin\lambda\cos\lambda \\ \beta\cos\lambda & 1-\frac{1}{2}\beta^2 & -\beta\sin\lambda \\ \frac{1}{2}\beta^2\sin\lambda\cos\lambda & \beta\sin\lambda & 1-\frac{1}{2}\beta^2\sin^2\lambda \end{pmatrix}. \tag{13}$$

The inclination angle γ_f of the front wheel can be found by transforming the local y-axis to the global frame and considering its z-component as $\gamma_\mathrm{f} = \phi + \beta\sin\lambda$. The centre of the front wheel has local coordinates in the front fork coordinate system $(-t, 0, -r_\mathrm{f})$. After transformation, its z-component is

$$\begin{aligned} z_{0\mathrm{f}} &= z - w\chi - r_\mathrm{f}(1-\tfrac{1}{2}\phi^2-\tfrac{1}{2}\beta^2\sin^2\lambda-\phi\beta\sin\lambda) - t\beta\cos\lambda(\phi+\tfrac{1}{2}\beta\sin\lambda) \\ &= -\rho_\mathrm{f} - (r_\mathrm{f}-\rho_\mathrm{f})(1-\tfrac{1}{2}\gamma_\mathrm{f}^2). \end{aligned} \tag{14}$$

With Eq. (12), this gives the pitch angle

$$\chi = \frac{1}{w}\big[-\tfrac{1}{2}\rho_{\rm r}\phi^2 + \tfrac{1}{2}\rho_{\rm f}(\phi+\beta\sin\lambda)^2 - t\beta\cos\lambda(\phi+\tfrac{1}{2}\beta\sin\lambda)\big] \tag{15}$$

The first two terms between the square brackets give the contributions due to the finite crown radii of the tires and the last term is the contribution known from the analysis with knife-edge wheels.

The positions of the other characteristic points can now easily be found. The rear frame assembly, the combined rear frame and the rear wheel, has mass $m_{\rm R}$ and its centre of mass has coordinates $(x_{\rm R}, 0, z_{\rm R})$, while the front fork assembly, the combined front fork and front wheel, has mass $m_{\rm F}$ and its centre of mass has coordinates $(x_{\rm F} - w - t, 0, z_{\rm F})$:

$$\begin{aligned} m_{\rm R} &= m_{\rm r} + m_{\rm rf},\\ m_{\rm F} &= m_{\rm f} + m_{\rm ff},\\ x_{\rm R} &= m_{\rm rf}x_{\rm rf}/m_{\rm R},\\ x_{\rm F} &= (m_{\rm f}w + m_{\rm ff}x_{\rm ff})/m_{\rm F},\\ z_{\rm R} &= (-m_{\rm r}r_{\rm r} + m_{\rm rf}z_{\rm rf})/m_{\rm R},\\ z_{\rm F} &= (-m_{\rm f}r_{\rm f} + m_{\rm ff}z_{\rm ff})/m_{\rm F}. \end{aligned} \tag{16}$$

The position of the centre of mass of the rear frame assembly becomes

$$\mathbf{x}_{\rm 0R} = \begin{pmatrix} x + (1-\frac{1}{2}\psi^2)x_{\rm R} + (\chi+\psi\phi)z_{\rm R} \\ y + \psi x_{\rm R} - \phi z_{\rm R} \\ z - \chi x_{\rm R} + (1-\frac{1}{2}\phi^2)z_{\rm R} \end{pmatrix}, \tag{17}$$

and likewise the position of the pressure point and the centre of the rear wheel are

$$\mathbf{x}_{\rm d} = \begin{pmatrix} x + (1-\frac{1}{2}\psi^2)x_{\rm d} + (\chi+\psi\phi)z_{\rm d} \\ y + \psi x_{\rm d} - \phi z_{\rm d} \\ z - \chi x_{\rm d} + (1-\frac{1}{2}\phi^2)z_{\rm d} \end{pmatrix}, \quad \mathbf{x}_{\rm 0r} = \begin{pmatrix} x - (\chi+\psi\phi)r_{\rm r} \\ y + \phi r_{\rm r} \\ z - (1-\frac{1}{2}\phi^2)r_{\rm r} \end{pmatrix}. \tag{18}$$

The centre of mass of the front fork assembly can be found as

$$\mathbf{x}_{\rm 0F} = \begin{pmatrix} x + (1-\frac{1}{2}\psi^2)x_{\rm F} + (\chi+\psi\phi)z_{\rm F} - \beta(\psi+\frac{1}{2}\beta\cos\lambda)u_{\rm F} \\ y + \psi x_{\rm F} - \phi z_{\rm F} + \beta u_{\rm F} \\ z - \chi x_{\rm F} + (1-\frac{1}{2}\phi^2)z_{\rm F} + \beta(\phi+\frac{1}{2}\beta\sin\lambda)u_{\rm F} \end{pmatrix}, \tag{19}$$

where $u_{\rm F} = (x_{\rm F} - w - t)\cos\lambda - z_{\rm F}\sin\lambda$ is the forward distance of the centre of mass of the front fork assembly to the steering axis. The position of the centre of the front wheel is

$$\mathbf{x}_{\rm 0f} = \begin{pmatrix} x + (1-\frac{1}{2}\psi^2)w - (\chi+\psi\phi)r_{\rm f} - \beta(\psi+\frac{1}{2}\beta\cos\lambda)u_{\rm f} \\ y + \psi w + \phi r_{\rm f} + \beta u_{\rm f} \\ z - \chi w - (1-\frac{1}{2}\phi^2)r_{\rm f} + \beta(\phi+\frac{1}{2}\beta\sin\lambda)u_{\rm f} \end{pmatrix}, \tag{20}$$

where $u_{\rm f} = -t\cos\lambda + r_{\rm f}\sin\lambda$ is the forward distance of the centre the front wheel to the steering axis.

3.2 Velocity analysis

The linear velocities expressed in components with respect to the global coordinate system can easily be found by differentiating the expressions for the positions with respect to time. The differentiation of Eq. (12) and (15) gives for the time derivatives of the dependent coordinates,

$$\dot{z} = -\rho_{\rm r}\phi\dot{\phi}, \tag{21}$$

$$\begin{aligned}\dot{\chi} &= \frac{1}{w}\big[[(\rho_{\rm f} - \rho_{\rm r})\phi + (\rho_{\rm f}\sin\lambda - t\cos\lambda)\beta]\dot{\phi} + (\rho_{\rm f}\sin\lambda - t\cos\lambda)(\phi + \beta\sin\lambda)\dot{\beta}\big] \\ &= (f_\rho\phi - f_{\rm m}\beta)\dot{\phi} - f_{\rm m}(\phi + \beta\sin\lambda)\dot{\beta},\end{aligned} \tag{22}$$

where

$$f_\rho = \frac{\rho_{\rm f} - \rho_{\rm r}}{w}, \quad f_{\rm m} = \frac{t\cos\lambda - \rho_{\rm f}\sin\lambda}{w}. \tag{23}$$

$f_{\rm m}$ is a kind of effective mechanical trail divided by the wheelbase. The angular velocities are most conveniently expressed in body-fixed components. For the rear frame assembly and the front fork assembly this yields respectively

$$\boldsymbol{\omega}_{\rm R} = \begin{pmatrix} \dot{\phi} \\ \phi\dot{\psi} + \dot{\chi} \\ \dot{\psi} \end{pmatrix}, \quad \boldsymbol{\omega}_{\rm F} = \begin{pmatrix} \dot{\phi} + \dot{\beta}\sin\lambda \\ (\phi + \beta\sin\lambda)\dot{\psi} - \beta\dot{\phi}\cos\lambda + \dot{\chi} \\ \dot{\psi} + \dot{\beta}\cos\lambda \end{pmatrix} \tag{24}$$

The angular velocities of the wheels are found by adding $\dot{\chi}_{\rm r}$ and $\dot{\chi}_{\rm f}$ to the respective second components.

The non-holonomic constraints at the wheels gives rise to dependencies in the velocities. First, the constraints at the rear wheel are analysed. The velocity of the centre of the rear wheel can be found by differentiating the expression in Eq. (18) with respect to time,

$$\dot{\mathbf{x}}_{0\rm r} = \begin{pmatrix} \dot{x} - r_{\rm r}(\dot{\chi} + \phi\dot{\psi} + \psi\dot{\phi}) \\ \dot{y} + r_{\rm r}\dot{\phi} \\ \dot{z} + r_{\rm r}\phi\dot{\phi} \end{pmatrix}. \tag{25}$$

It is convenient to transform these components into the directions in which the longitudinal and lateral slips are defined, that is, for the rear wheel, the directions parallel and normal to the line of intersection of the plane of symmetry with the road surface. This transformation only involves the yaw angle ψ, and yields the velocity and angular velocity of the rear wheel

$$\dot{\mathbf{x}}^{\rm c}_{0\rm r} = \begin{pmatrix} \dot{x} - r_{\rm r}(\dot{\chi} + \phi\dot{\psi}) + \psi\dot{y} \\ -\psi\dot{x} + \dot{y} + r_{\rm r}\dot{\phi} \\ \dot{z} + r_{\rm r}\phi\dot{\phi} \end{pmatrix}, \quad \boldsymbol{\omega}^{\rm c}_{\rm r} = \begin{pmatrix} \dot{\phi} \\ \dot{\chi} + \dot{\chi}_{\rm r} \\ \dot{\psi} + \phi\dot{\chi}_{\rm r} \end{pmatrix}. \tag{26}$$

With the position of the contact point with respect to the wheel centre $(0, -(r_{\rm r} - \rho_{\rm r})\phi, r_{\rm r})$, the velocity of the contact point becomes

$$\dot{\mathbf{x}}_{\rm rc} = \begin{pmatrix} \dot{x} + r_{\rm r}\dot{\chi}_{\rm r} - \rho_{\rm r}\phi\dot{\psi} + \psi\dot{y} \\ -\psi\dot{x} + \dot{y} \\ \dot{z} + \rho_{\rm r}\phi\dot{\phi} \end{pmatrix}. \tag{27}$$

Because of Eq. (21), the z-component is identically equal to zero. The normal spin rate is $-\dot{\psi} - \phi\dot{\chi}_{\rm r} = -\dot{\psi} + \phi v/r_{\rm r}$, which yields the constraint for the lateral velocity,

$$-\psi\dot{x} + \dot{y} + t_{\rm pr}(-\dot{\psi} - \phi\dot{\chi}_{\rm r}) = 0, \quad \dot{y} = (\psi - \frac{t_{\rm pr}}{r_{\rm r}}\phi)v + t_{\rm pr}\dot{\psi}. \tag{28}$$

The constraint in the longitudinal direction then yields

$$\dot{x} = -r_{\rm r}\dot{\chi}_{\rm r} + (\rho_{\rm r}\phi - t_{\rm pr}\psi)\dot{\psi} = v + (\rho_{\rm r}\phi - t_{\rm pr}\psi)\dot{\psi}. \tag{29}$$

For the front wheel, the local contact coordinate system is rotated with respect to the global frame about the z-axis by an angle $\psi + \beta\cos\lambda$. The velocity of the wheel centre in the global coordinate system can be found by differentiating Eq. (20) as

$$\dot{\mathbf{x}}_{0\rm f} = \begin{pmatrix} \dot{x} - r_{\rm f}\dot{\chi} - (w\psi + r_{\rm f}\phi + u_{\rm f}\beta)\dot{\psi} - r_{\rm f}\psi\dot{\phi} - u_{\rm f}(\psi + \beta\cos\lambda)\dot{\beta} \\ \dot{y} + w\dot{\psi} + r_{\rm f}\dot{\phi} + u_{\rm f}\dot{\beta} \\ \dot{z} - w\dot{\chi} + (r_{\rm f}\phi + u_{\rm f}\beta)\dot{\phi} + u_{\rm f}(\phi + \beta\sin\lambda)\dot{\beta} \end{pmatrix}. \tag{30}$$

The velocity of the centre of the wheel and its angular velocity in the rotated frame becomes

$$\dot{\mathbf{x}}_{0\rm f}^{\rm c} = \begin{pmatrix} \dot{x} + (\psi + \beta\cos\lambda)\dot{y} - r_{\rm f}\dot{\chi} - [r_{\rm f}\phi + (u_{\rm f} - w\cos\lambda)\beta]\dot{\psi} + r_{\rm f}\beta\cos\lambda\dot{\phi} \\ -(\psi + \beta\cos\lambda)\dot{x} + \dot{y} + w\dot{\psi} + r_{\rm f}\dot{\phi} + u_{\rm f}\dot{\beta} \\ \dot{z} - w\dot{\chi} + (r_{\rm f}\phi + u_{\rm f}\beta)\dot{\phi} + u_{\rm f}(\phi + \beta\sin\lambda)\dot{\beta} \end{pmatrix}. \tag{31}$$

The angular velocity of the front wheel can be transformed to the same frame as

$$\boldsymbol{\omega}_{\rm f}^{\rm c} = \begin{pmatrix} \dot{\phi} + \dot{\beta}\sin\lambda \\ \dot{\chi} + \dot{\chi}_{\rm f} - \beta\dot{\phi}\cos\lambda - (\phi + \beta\sin\lambda)\dot{\beta}\cos\lambda \\ \dot{\psi} + \dot{\beta}\cos\lambda + (\phi + \beta\sin\lambda)\dot{\chi}_{\rm f} \end{pmatrix}. \tag{32}$$

With the coordinates of the contact point with respect to the wheel centre, $(0, -(r_{\rm f} - \rho_{\rm f})(\phi + \beta\sin\lambda), r_{\rm f})$, the velocity of the contact point becomes

$$\mathbf{v}_{\rm fc} = \begin{pmatrix} \dot{x} + (\psi + \beta\cos\lambda)\dot{y} + r_{\rm f}\dot{\chi}_{\rm f} + (w + t)\beta\dot{\psi}\cos\lambda - \rho_{\rm f}(\phi + \beta\sin\lambda)(\dot{\psi} + \dot{\beta}\cos\lambda) \\ -(\psi + \beta\cos\lambda)\dot{x} + \dot{y} + w\dot{\psi} - t\dot{\beta}\cos\lambda \\ \dot{z} - w\dot{\chi} + [-t\beta\cos\lambda + \rho_{\rm f}(\phi + \beta\sin\lambda)]\dot{\phi} + (\phi + \beta\sin\lambda)(-t\cos\lambda + \rho_{\rm f}\sin\lambda)\dot{\beta} \end{pmatrix}. \tag{33}$$

By substituting the time derivatives of Eq. (12) and (15), it is seen that the speed in the normal direction is zero, as it should be. From the lateral slip and the normal spin rate $-\dot{\psi} - \dot{\beta}\cos\lambda - (\phi + \beta\sin\lambda)\dot{\chi}_{\rm f}$, it follows that the yaw rate is

$$\begin{aligned} \dot{\psi} &= \frac{1}{t_{\rm pr} + w - t_{\rm pf}}\Big[\big[\beta\cos\lambda + \frac{t_{\rm pr}}{r_{\rm r}}\phi - (\phi + \beta\sin\lambda)\frac{t_{\rm pf}}{r_{\rm f}}\big]v + (t + t_{\rm pf})\dot{\beta}\cos\lambda\Big] \\ &= \big[f_\phi\phi + f_\beta\beta\big]v + f\dot{\beta}, \end{aligned} \tag{34}$$

where

$$f = \frac{(t + t_{\rm pf})\cos\lambda}{t_{\rm pr} + w - t_{\rm pf}}, \quad f_\phi = \frac{\dfrac{t_{\rm pr}}{r_{\rm r}} - \dfrac{t_{\rm pf}}{r_{\rm f}}}{t_{\rm pr} + w - t_{\rm pf}}, \quad f_\beta = \frac{\cos\lambda - \dfrac{t_{\rm pf}}{r_{\rm f}}\sin\lambda}{t_{\rm pr} + w - t_{\rm pf}}. \tag{35}$$

From this equation, it appears that for the lateral motion, the wheelbase is effectively modified to $t_{\rm pr} + w - t_{\rm pf}$ and the trail is effectively increased to $t + t_{\rm pf}$. From the condition of zero longitudinal slip, it follows that the rotation rate of the front wheel is

$$\dot\chi_{\rm f} = \frac{-1}{r_{\rm f}}\Big[\dot x + \big[\beta(w+t)\cos\lambda + t_{\rm pr}(\psi+\beta\cos\lambda)\big]\dot\psi - \rho_{\rm f}(\phi+\beta\sin\lambda)(\dot\psi+\dot\beta\cos\lambda)\Big]. \quad (36)$$

With the known relations for the dependent velocities, the velocity of the centre of mass of the rear frame assembly can be found by differentiating the position in Eq. (17),

$$\dot{\mathbf{x}}_{\rm 0R} = \begin{pmatrix} \dot x - x_{\rm R}\psi\dot\psi + z_{\rm R}(\dot\chi + \phi\dot\psi + \psi\dot\phi) \\ \dot y + x_{\rm R}\dot\psi - z_{\rm R}\dot\phi \\ \dot z - x_{\rm R}\dot\chi - z_{\rm R}\phi\dot\phi \end{pmatrix}, \quad (37)$$

and likewise for the velocity of the pressure point by taking derivatives of Eq. (18),

$$\dot{\mathbf{x}}_{\rm d} = \begin{pmatrix} \dot x - x_{\rm d}\psi\dot\psi + z_{\rm d}(\dot\chi + \phi\dot\psi + \psi\dot\phi) \\ \dot y + x_{\rm d}\dot\psi - z_{\rm d}\dot\phi \\ \dot z - x_{\rm d}\dot\chi - z_{\rm d}\phi\dot\phi \end{pmatrix}. \quad (38)$$

The velocity of the centre of mass of the front frame is found by differentiating Eq. (19),

$$\dot{\mathbf{x}}_{\rm 0F} = \begin{pmatrix} \dot x - x_{\rm F}\psi\dot\psi + z_{\rm F}(\dot\chi + \phi\dot\psi + \psi\dot\phi) - u_{\rm F}(\beta\dot\psi + \psi\dot\beta) - u_{\rm F}\beta\dot\beta\cos\lambda \\ \dot y + x_{\rm F}\dot\psi - z_{\rm F}\dot\phi + u_{\rm F}\dot\beta \\ \dot z - x_{\rm F}\dot\chi - z_{\rm F}\phi\dot\phi + u_{\rm F}(\beta\dot\phi + \phi\dot\beta) + u_{\rm F}\beta\dot\beta\sin\lambda \end{pmatrix}. \quad (39)$$

3.3 Virtual displacements

The virtual displacements are now easily obtained from the expressions of the velocities: because the system is scleronomic, the time derivatives only need to be replaced by virtual displacements. This yields the relations between the virtual variations,

$$\begin{aligned}
&\delta z = -\rho_{\rm r}\phi\delta\phi, \\
&\delta\chi = (f_\rho\phi - f_{\rm m}\beta)\delta\phi - f_{\rm m}(\phi+\beta\sin\lambda)\delta\beta, \\
&\delta\psi = -\big[f_\phi\phi + f_\beta\beta\big]r_{\rm r}\delta\chi_{\rm r} + f\delta\beta, \\
&\delta x = -r_{\rm r}\delta\chi_{\rm r} + (\rho_{\rm r}\phi - t_{\rm pr}\psi)\delta\psi, \\
&\delta y = -(r_{\rm r}\psi - t_{\rm pr}\phi)\delta\chi_{\rm r} + t_{\rm pr}\delta\psi, \\
&\delta\chi_{\rm f} = \frac{r_{\rm r}}{r_{\rm f}}\delta\chi_{\rm r} - \frac{1}{r_{\rm f}}\Big[\big[\beta(t_{\rm pr}+w+t)\cos\lambda - \rho_{\rm f}\beta\sin\lambda + (\rho_{\rm r}-\rho_{\rm f})\phi\big]\delta\psi \\
&\qquad -\rho_{\rm f}(\phi+\beta\sin\lambda)\delta\beta\cos\lambda\Big].
\end{aligned} \quad (40)$$

The virtual displacement of the centre of mass of the rear frame assembly is

$$\delta\mathbf{x}_{\rm 0R} = \begin{pmatrix} \delta x - x_{\rm R}\psi\delta\psi + z_{\rm R}(\delta\chi + \phi\delta\psi + \psi\delta\phi) \\ \delta y + x_{\rm R}\delta\psi - z_{\rm R}\delta\phi \\ \delta z - x_{\rm R}\delta\chi - z_{\rm R}\phi\delta\phi \end{pmatrix}, \quad (41)$$

and likewise for the virtual displacement of the pressure point is

$$\delta\mathbf{x}_{\rm d} = \begin{pmatrix} \delta x - x_{\rm d}\psi\delta\psi + z_{\rm d}(\delta\chi + \phi\delta\psi + \psi\delta\phi) \\ \delta y + x_{\rm d}\delta\psi - z_{\rm d}\delta\phi \\ \delta z - x_{\rm d}\delta\chi - z_{\rm d}\phi\delta\phi \end{pmatrix}. \quad (42)$$

The virtual displacement of the centre of mass of the front frame is

$$\delta\mathbf{x}_{0\mathrm{F}} = \begin{pmatrix} \delta x - x_{\mathrm{F}}\psi\delta\psi + z_{\mathrm{F}}(\delta\chi + \phi\delta\psi + \psi\delta\phi) - u_{\mathrm{F}}\beta\delta\beta\cos\lambda - u_{\mathrm{F}}(\beta\delta\psi + \psi\delta\beta) \\ \delta y + x_{\mathrm{F}}\delta\psi - z_{\mathrm{F}}\delta\phi + u_{\mathrm{F}}\delta\beta \\ \delta z - x_{\mathrm{F}}\delta\chi - z_{\mathrm{F}}\phi\delta\phi + u_{\mathrm{F}}(\beta\delta\phi + \phi\delta\beta) + u_{\mathrm{F}}\beta\delta\beta\sin\lambda \end{pmatrix}. \quad (43)$$

The virtual rotations of the rear frame assembly and the front fork assembly are

$$\delta\boldsymbol{\phi}_{\mathrm{R}} = \begin{pmatrix} \delta\phi \\ \phi\delta\psi + \delta\chi \\ \delta\psi \end{pmatrix}, \quad \delta\boldsymbol{\phi}_{\mathrm{F}} = \begin{pmatrix} \delta\phi + \delta\beta\sin\lambda \\ (\phi + \beta\sin\lambda)\delta\psi - \beta\delta\phi\cos\lambda + \delta\chi \\ \delta\psi + \delta\beta\cos\lambda \end{pmatrix}. \quad (44)$$

3.4 Accelerations

The accelerations are found by taking a further derivative with respect to time of the velocities and angular velocities. There is no need to retain the second-order terms, so the expressions simplify considerably. The relations for the dependent accelerations are found from differentiating the expressions already found for the velocities; this yields

$$\begin{aligned} \ddot{z} &= 0, \\ \ddot{\chi} &= 0, \\ \ddot{x} &= \dot{v}, \\ \ddot{y} &= (\psi - \frac{t_{\mathrm{pr}}}{r_{\mathrm{r}}}\phi)\dot{v} + (\dot{\psi} - \frac{t_{\mathrm{pr}}}{r_{\mathrm{r}}}\dot{\phi})v + t_{\mathrm{pr}}\ddot{\psi}, \\ \ddot{\psi} &= \left[f_\phi\phi + f_\beta\beta\right]\dot{v} + \left[f_\phi\dot{\phi} + f_\beta\dot{\beta}\right]v + f\ddot{\beta}, \\ \ddot{\chi}_{\mathrm{f}} &= \frac{-\dot{v}}{r_{\mathrm{f}}} \end{aligned} \quad (45)$$

The acceleration of the centre of mass of the rear frame is

$$\ddot{\mathbf{x}}_{0\mathrm{R}} = \begin{pmatrix} \dot{v} \\ \ddot{y} + x_{\mathrm{R}}\ddot{\psi} - z_{\mathrm{R}}\ddot{\phi} \\ 0 \end{pmatrix}, \quad (46)$$

The acceleration of the centre of mass of the front frame assembly is found to be

$$\ddot{\mathbf{x}}_{0\mathrm{F}} = \begin{pmatrix} \dot{v} \\ \ddot{y} + x_{\mathrm{F}}\ddot{\psi} - z_{\mathrm{F}}\ddot{\phi} + u_{\mathrm{F}}\ddot{\beta} \\ 0 \end{pmatrix}. \quad (47)$$

The angular accelerations of the rear frame assembly and the front fork assembly are equal to the time derivatives of the angular velocities in Eq. (24),

$$\dot{\boldsymbol{\omega}}_{\mathrm{R}} = \begin{pmatrix} \ddot{\phi} \\ 0 \\ \ddot{\psi} \end{pmatrix}, \quad \dot{\boldsymbol{\omega}}_{\mathrm{F}} = \begin{pmatrix} \ddot{\phi} + \ddot{\beta}\sin\lambda \\ 0 \\ \ddot{\psi} + \ddot{\beta}\cos\lambda \end{pmatrix}. \quad (48)$$

3.5 Aerodynamic drag force

The aerodynamic drag force, in contrast to the gravity force, does not have a constant magnitude and direction. Due to the lateral velocity of the pressure point, a lateral component is generated,

$$F_{\mathrm{d}y} = -F_{\mathrm{d}}(\dot{y} + x_{\mathrm{d}}\dot{\psi} - z_{\mathrm{d}}\dot{\phi})/v\,. \quad (49)$$

The longitudinal component is unchanged and the vertical component is zero in a linear approximation.

3.6 Virtual work

The equations of motion are derived by collecting all contributions to the virtual work. The terms involving the linear accelerations are easily obtained from

$$-m_{\mathrm{R}}\delta\mathbf{x}_{\mathrm{R}}^T\ddot{\mathbf{x}}_{\mathrm{R}} - m_{\mathrm{F}}\delta\mathbf{x}_{\mathrm{F}}^T\ddot{\mathbf{x}}_{\mathrm{F}}. \tag{50}$$

The contribution of the rotational part is

$$\begin{aligned} &-\delta\boldsymbol{\phi}_{\mathrm{R}}^T\big[\mathbf{I}_{\mathrm{R}}\dot{\boldsymbol{\omega}}_{\mathrm{R}} + I_{\mathrm{r}yy}\ddot{\chi}_{\mathrm{r}}\mathbf{e}_y + \boldsymbol{\omega}_{\mathrm{R}} \times \big(\mathbf{I}_{\mathrm{R}}\boldsymbol{\omega}_{\mathrm{R}} + I_{\mathrm{r}yy}\dot{\chi}_{\mathrm{r}}\mathbf{e}_y\big)\big] - \delta\chi_{\mathrm{r}}I_{\mathrm{r}yy}\ddot{\chi}_{\mathrm{r}} \\ &\quad -\delta\boldsymbol{\phi}_{\mathrm{F}}^T\big[\mathbf{I}_{\mathrm{F}}\dot{\boldsymbol{\omega}}_{\mathrm{F}} + I_{\mathrm{f}yy}\ddot{\chi}_{\mathrm{f}}\mathbf{e}_y + \boldsymbol{\omega}_{\mathrm{F}} \times \big(\mathbf{I}_{\mathrm{F}}\boldsymbol{\omega}_{\mathrm{F}} + I_{\mathrm{f}yy}\dot{\chi}_{\mathrm{f}}\mathbf{e}_y\big)\big] - \delta\chi_{\mathrm{f}}I_{\mathrm{f}yy}\ddot{\chi}_{\mathrm{f}}. \end{aligned} \tag{51}$$

Here, $\mathbf{e}_y$ is the second unit vector and $\mathbf{I}_{\mathrm{R}}$ and $\mathbf{I}_{\mathrm{F}}$ are the moment of inertia tensors of the rear frame assembly and the front fork assembly,

$$\mathbf{I}_{\mathrm{R}} = \begin{pmatrix} I_{\mathrm{R}xx} & 0 & I_{\mathrm{R}xz} \\ 0 & I_{\mathrm{R}yy} & 0 \\ I_{\mathrm{R}xz} & 0 & I_{\mathrm{R}zz} \end{pmatrix}, \quad \mathbf{I}_{\mathrm{F}} = \begin{pmatrix} I_{\mathrm{F}xx} & 0 & I_{\mathrm{F}xz} \\ 0 & I_{\mathrm{R}yy} & 0 \\ I_{\mathrm{F}xz} & 0 & I_{\mathrm{F}zz} \end{pmatrix}, \tag{52}$$

with the components

$$\begin{aligned} I_{\mathrm{R}xx} &= I_{\mathrm{r}xx} + I_{\mathrm{rf}xx} + m_{\mathrm{r}}(r_{\mathrm{r}} + z_{\mathrm{R}})^2 + m_{\mathrm{rf}}(z_{\mathrm{rf}} - z_{\mathrm{R}})^2, \\ I_{\mathrm{R}xz} &= I_{\mathrm{rf}xz} - m_{\mathrm{r}}(r_{\mathrm{r}} + z_{\mathrm{R}})x_{\mathrm{R}} - m_{\mathrm{rf}}(z_{\mathrm{rf}} - z_{\mathrm{R}})(x_{\mathrm{rf}} - x_{\mathrm{R}}), \\ I_{\mathrm{R}yy} &= I_{\mathrm{r}yy} + I_{\mathrm{rf}yy} + m_{\mathrm{r}}[(r_{\mathrm{r}} + z_{\mathrm{R}})^2 + x_{\mathrm{R}}^2] + m_{\mathrm{rf}}[(z_{\mathrm{rf}} - z_{\mathrm{R}})^2 + (x_{\mathrm{rf}} - x_{\mathrm{R}})^2], \\ I_{\mathrm{R}zz} &= I_{\mathrm{r}zz} + I_{\mathrm{rf}zz} + m_{\mathrm{r}}x_{\mathrm{R}}^2 + m_{\mathrm{rf}}(x_{\mathrm{rf}} - x_{\mathrm{R}})^2, \\ I_{\mathrm{F}xx} &= I_{\mathrm{f}xx} + I_{\mathrm{ff}xx} + m_{\mathrm{f}}(r_{\mathrm{f}} + z_{\mathrm{F}})^2 + m_{\mathrm{ff}}(z_{\mathrm{ff}} - z_{\mathrm{F}})^2, \\ I_{\mathrm{F}xz} &= I_{\mathrm{ff}xz} + m_{\mathrm{f}}(r_{\mathrm{f}} + z_{\mathrm{F}})(w - x_{\mathrm{F}}) - m_{\mathrm{ff}}(z_{\mathrm{ff}} - z_{\mathrm{F}})(x_{\mathrm{ff}} - x_{\mathrm{F}}), \\ I_{\mathrm{F}yy} &= I_{\mathrm{f}yy} + I_{\mathrm{ff}yy} + m_{\mathrm{f}}[(r_{\mathrm{f}} + z_{\mathrm{F}})^2 + (w - x_{\mathrm{F}})^2] + m_{\mathrm{ff}}[(z_{\mathrm{ff}} - z_{\mathrm{F}})^2 + (x_{\mathrm{ff}} - x_{\mathrm{F}})^2], \\ I_{\mathrm{F}zz} &= I_{\mathrm{f}zz} + I_{\mathrm{ff}zz} + m_{\mathrm{f}}(w - x_{\mathrm{F}})^2 + m_{\mathrm{ff}}(x_{\mathrm{ff}} - x_{\mathrm{F}})^2. \end{aligned} \tag{53}$$

The gravity forces give a contribution

$$(m_{\mathrm{R}}\delta\mathbf{x}_{\mathrm{R}} + m_{\mathrm{F}}\delta\mathbf{x}_{\mathrm{F}})^T(g_x\mathbf{e}_x + g_z\mathbf{e}_z), \tag{54}$$

where e_x is the first and e_z is the third unit vector. The aerodynamic drag force gives a contribution

$$\delta\mathbf{x}_{\mathrm{d}}(-F_{\mathrm{d}}\mathbf{e}_x + F_{\mathrm{d}y}\mathbf{e}_y). \tag{55}$$

The driving or braking moments give a contribution

$$-M_{\mathrm{r}}\delta\chi_{\mathrm{r}} - M_{\mathrm{f}}\delta\chi_{\mathrm{f}}. \tag{56}$$

Finally, the aligning moments in the contact points give a contribution

$$\delta\psi C_{y\mathrm{r}}t_{\mathrm{pr}}^2\big(-\dot{\psi} + \frac{\phi v}{r_{\mathrm{r}}}\big)/v + (\delta\psi + \delta\beta\cos\lambda)C_{y\mathrm{f}}t_{\mathrm{pf}}^2\big[-\dot{\psi} - \dot{\beta}\cos\lambda + (\phi + \beta\sin\lambda)\frac{v}{r_{\mathrm{f}}}\big]/v. \tag{57}$$

3.7 Equations of motion

The equations of motion are now found by collecting all terms and substituting the expressions for the dependent coordinates and velocities. For a variation $\delta\chi_{\rm r}$, the non-linear equation (6) is retrieved. For the derivation of the lateral equations, the variation $\delta\chi_{\rm r}$ can be dropped. Initially, $\delta\psi$ is considered to be an independent variation and its relation to $\delta\beta$ is only imposed later on. The large terms multiplied by small virtual variations appear in the x- and z-direction due to the gravity force, the acceleration and the drag force. For the moments, only the angular acceleration of the wheels gives a contribution. In the lateral direction, only the dependent variations δy and $\delta\psi$ need be considered, as well as the independent variations $\delta\phi$ and $\delta\beta$. Some auxiliary quantities are defined as follows.

$$
\begin{aligned}
&I_{{\rm T}xx} = I_{{\rm R}xx} + I_{{\rm F}xx} + m_{\rm R} z_{\rm R}^2 + m_{\rm F} z_{\rm F}^2,\\
&I_{{\rm T}xz} = I_{{\rm R}xz} + I_{{\rm F}xz} - m_{\rm R}(t_{\rm pr}+x_{\rm R})z_{\rm R} - m_{\rm F}(t_{\rm pr}+x_{\rm F})z_{\rm F},\\
&I_{{\rm T}zz} = I_{{\rm R}zz} + I_{{\rm F}zz} + m_{\rm R}(t_{\rm pr}+x_{\rm R})^2 + m_{\rm F}(t_{\rm pr}+x_{\rm F})^2,\\
&I_{{\rm F}\lambda\lambda} = I_{{\rm F}xx}\sin^2\lambda + 2I_{{\rm F}xz}\sin\lambda\cos\lambda + I_{{\rm F}zz}\cos^2\lambda + m_{\rm F}u_{\rm F}^2,\\
&I_{{\rm F}x\lambda} = I_{{\rm F}xx}\sin\lambda + I_{{\rm F}xz}\cos\lambda - m_{\rm F}z_{\rm F}u_{\rm F},\\
&I_{{\rm F}z\lambda} = I_{{\rm F}xz}\sin\lambda + I_{{\rm F}zz}\cos\lambda + m_{\rm F}(t_{\rm pr}+x_{\rm F})u_{\rm F},\\
&S_x = m_{\rm R}z_{\rm R} + m_{\rm F}z_{\rm F},\\
&S_x' = m_{\rm R}(\rho_{\rm r} + f_\rho x_{\rm R} + z_{\rm R}) + m_{\rm F}(\rho_{\rm r} + f_\rho x_{\rm F} + z_{\rm F}),\\
&S_z = m_{\rm R}(t_{\rm pr}+x_{\rm R}) + m_{\rm F}(t_{\rm pr}+x_{\rm F}),\\
&S_z' = m_{\rm R}x_{\rm R} + m_{\rm F}x_{\rm F},\\
&S_\lambda = m_{\rm F}u_{\rm F},\\
&S_{\rm r} = I_{{\rm r}yy}/r_{\rm r},\\
&S_{\rm f} = I_{{\rm f}yy}/r_{\rm f},\\
&S_{\rm w} = S_{\rm r} + S_{\rm f},\\
&\bar C_{\rm d} = \tfrac{1}{2}\rho_{\rm air}C_{\rm d}A,\\
&\bar C_{y{\rm r}} = C_{y{\rm r}}t_{\rm pr}^2,\\
&\bar C_{y{\rm f}} = C_{y{\rm f}}t_{\rm pf}^2.
\end{aligned}
\tag{58}
$$

The unreduced equations have the form

$$
\bar{\mathbf M}\ddot{\bar{\mathbf q}} + (v\bar{\mathbf C}_1 + \bar{\mathbf C}_{-1}/v)\dot{\bar{\mathbf q}} + (\bar{\mathbf K}_0 + \dot v\bar{\mathbf K}_1 + v^2\bar{\mathbf K}_2)\bar{\mathbf q} = \mathbf 0. \tag{59}
$$

Here, $\bar{\mathbf q} = (\phi, \beta, \psi)^T$ contains the non-cyclic coordinates for the lateral motion and the matrices are

$$
\bar{\mathbf M} = \begin{pmatrix} I_{{\rm T}xx} & I_{{\rm F}x\lambda} & I_{{\rm T}xz} \\ I_{{\rm F}x\lambda} & I_{{\rm F}\lambda\lambda} & I_{{\rm F}z\lambda} \\ I_{{\rm T}xz} & I_{{\rm F}z\lambda} & I_{{\rm T}zz} \end{pmatrix}, \tag{60}
$$

$$
\bar{\mathbf C}_1 = \begin{pmatrix} S_x t_{\rm pr}/r_{\rm r} + \bar C_{\rm d} z_{\rm d}^2 & S_{\rm f}\cos\lambda & -S_x + S_{\rm w} - \bar C_{\rm d} z_{\rm d}(t_{\rm pr}+x_{\rm d}) \\ -S_\lambda t_{\rm pr}/r_{\rm r} - S_{\rm f}\cos\lambda & 0 & S_\lambda + S_{\rm f}\sin\lambda \\ -S_z t_{\rm pr}/r_{\rm r} - S_{\rm w} - \bar C_{\rm d} z_{\rm d}(t_{\rm pr}+x_{\rm d}) & -S_{\rm f}\sin\lambda & S_z + \bar C_{\rm d}(t_{\rm pr}+x_{\rm d})^2 \end{pmatrix}, \tag{61}
$$

$$
\bar{\mathbf C}_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar C_{y{\rm f}}\cos^2\lambda & \bar C_{y{\rm f}}\cos\lambda \\ 0 & \bar C_{y{\rm f}}\cos\lambda & \bar C_{y{\rm r}} + \bar C_{y{\rm f}} \end{pmatrix}. \tag{62}
$$

The stiffness matrices are given in their components.

$$
\begin{aligned}
&\bar{K}_{0,\phi\phi} = -f_\rho S_x g_x + S'_x g_z,\\
&\bar{K}_{0,\phi\beta} = f_\mathrm{m} S_x g_x - (f_\mathrm{m} S'_z + S_\lambda) g_z,\\
&\bar{K}_{0,\phi\psi} = -S_x g_x,\\
&\bar{K}_{0,\beta\phi} = f_\mathrm{m} S_x g_x - (f_\mathrm{m} S'_z + S_\lambda) g_z + M_\mathrm{f}\rho_\mathrm{f}\cos\lambda/r_\mathrm{f} - \bar{C}_{y\mathrm{f}}\cos\lambda/r_\mathrm{f},\\
&\bar{K}_{0,\beta\beta} = (f_\mathrm{m} S_x \sin\lambda + S_\lambda\cos\lambda) g_x - (f_\mathrm{m} S'_z + S_\lambda)\sin\lambda g_z\\
&\qquad + M_\mathrm{f}\rho_\mathrm{f}\sin\lambda\cos\lambda/r_\mathrm{f} - \bar{C}_{y\mathrm{f}}\sin\lambda\cos\lambda/r_\mathrm{f},\\
&\bar{K}_{0,\beta\psi} = S_\lambda g_x,\\
&\bar{K}_{0,\psi\phi} = -(S_x + m_\mathrm{T}\rho_\mathrm{r}) g_x + M_\mathrm{f}(\rho_\mathrm{f} - \rho_\mathrm{r})/r_\mathrm{f} - \bar{C}_{y\mathrm{r}}/r_\mathrm{r} - \bar{C}_{y\mathrm{f}}/r_\mathrm{f},\\
&\bar{K}_{0,\psi\beta} = S_\lambda g_x + M_\mathrm{f}[\rho_\mathrm{f}\sin\lambda - (t_\mathrm{pr} + w + t)\cos\lambda]/r_\mathrm{f} - \bar{C}_{y\mathrm{f}}\sin\lambda/r_\mathrm{f},\\
&\bar{K}_{0,\psi\psi} = S_z g_x;
\end{aligned}
\qquad (63)
$$

$$
\begin{aligned}
&\bar{K}_{1,\phi\phi} = (f_\rho + t_\mathrm{pr}/r_\mathrm{r}) S_x - f_\rho S_\mathrm{w},\\
&\bar{K}_{1,\phi\beta} = -f_\mathrm{m}(S_x - S_\mathrm{w}) + S_\mathrm{f}\cos\lambda,\\
&\bar{K}_{1,\phi\psi} = 0,\\
&\bar{K}_{1,\beta\phi} = -f_\mathrm{m}(S_x - S_\mathrm{w}) - S_\lambda t_\mathrm{pr}/r_\mathrm{r} - S_\mathrm{f}\rho_\mathrm{f}\cos\lambda/r_\mathrm{f},\\
&\bar{K}_{1,\beta\beta} = -f_\mathrm{m}(S_x - S_\mathrm{w})\sin\lambda - S_\lambda\cos\lambda - S_\mathrm{f}\rho_\mathrm{f}\sin\lambda\cos\lambda/r_\mathrm{f},\\
&\bar{K}_{1,\beta\psi} = 0,\\
&\bar{K}_{1,\psi\phi} = S_x + m_\mathrm{T}\rho_\mathrm{r} - S_z t_\mathrm{pr}/r_\mathrm{r} - (\rho_\mathrm{f} - \rho_\mathrm{r}) S_\mathrm{f}/r_\mathrm{f} - S_\mathrm{w},\\
&\bar{K}_{1,\psi\beta} = -S_\lambda - S_\mathrm{f}[(r_\mathrm{f} + \rho_\mathrm{f})\sin\lambda - (t_\mathrm{pr} + w + t)\cos\lambda]/r_\mathrm{f},\\
&\bar{K}_{1,\psi\psi} = 0;
\end{aligned}
\qquad (64)
$$

$$
\bar{\mathbf{K}}_2 = \begin{pmatrix} \bar{C}_\mathrm{d} z_\mathrm{d}(f_\rho + t_\mathrm{pr}/r_\mathrm{r}) & -\bar{C}_\mathrm{d} z_\mathrm{d} f_\mathrm{m} & 0 \\ -\bar{C}_\mathrm{d} z_\mathrm{d} f_\mathrm{m} & -\bar{C}_\mathrm{d} z_\mathrm{d} f_\mathrm{m}\sin\lambda & 0 \\ \bar{C}_\mathrm{d}[\rho_\mathrm{r} + z_\mathrm{d} - (t_\mathrm{pr} + x_\mathrm{d}) t_\mathrm{pr}/r_\mathrm{r}] & 0 & 0 \end{pmatrix}. \qquad (65)
$$

The expressions for $\dot{\psi}$ in Eq. (34) and $\ddot{\psi}$ in Eq. (45) can now be substituted and the relation for $\delta\psi$ in Eq. (40) be used by adding f times the third equation to the second equation in order to obtain the final equations in the form

$$
\mathbf{M}\ddot{\mathbf{q}}^\mathrm{d} + (\dot{x}\mathbf{C}_1 + \mathbf{C}_{-1}/\dot{x})\dot{\mathbf{q}}^\mathrm{d} + (\mathbf{K}_0 + \ddot{x}\mathbf{K}_1 + \dot{x}^2\mathbf{K}_2)\mathbf{q}^\mathrm{d} + \mathbf{K}^\mathrm{k}\mathbf{q}^\mathrm{k} = \mathbf{0}, \qquad (66)
$$

where $\mathbf{q}^\mathrm{d} = (\phi, \beta)^T$ are the dynamic degrees of freedom and $\mathbf{q}^\mathrm{k} = (\psi)$ is the non-cyclic kinematic coordinate. The mass matrix is

$$
\mathbf{M} = \begin{pmatrix} I_{\mathrm{T}xx} & I_{\mathrm{F}x\lambda} + f I_{\mathrm{T}xz} \\ I_{\mathrm{F}x\lambda} + f I_{\mathrm{T}xz} & I_{\mathrm{F}\lambda\lambda} + 2f I_{\mathrm{F}z\lambda} + f^2 I_{\mathrm{T}zz} \end{pmatrix}. \qquad (67)
$$

The entries in the damping matrices are

$$
\begin{aligned}
&C_{1,\phi\phi} = S_x t_\mathrm{pr}/r_\mathrm{r} + \bar{C}_\mathrm{d} z_\mathrm{d}^2 + f_\phi I_{\mathrm{T}xz},\\
&C_{1,\phi\beta} = S_\mathrm{f}\cos\lambda - f(S_x - S_\mathrm{w}) - f\bar{C}_\mathrm{d} z_\mathrm{d}(t_\mathrm{pr} + x_\mathrm{d}) + f_\beta I_{\mathrm{T}xz},\\
&C_{1,\beta\phi} = -S_\lambda t_\mathrm{pr}/r_\mathrm{r} - S_\mathrm{f}\cos\lambda - f S_z t_\mathrm{pr}/r_\mathrm{r} - f S_\mathrm{w} - f\bar{C}_\mathrm{d} z_\mathrm{d}(t_\mathrm{pr} + x_\mathrm{d}) + f_\phi(I_{\mathrm{F}z\lambda} + f I_{\mathrm{T}zz}),\\
&C_{1,\beta\beta} = f S_\lambda + f^2 S_z + f^2\bar{C}_\mathrm{d}(t_\mathrm{pr} + x_\mathrm{d})^2 + f_\beta(I_{\mathrm{F}z\lambda} + f I_{\mathrm{T}zz}),
\end{aligned}
\qquad (68)
$$

and

$$
\begin{aligned}
&C_{-1,\phi\phi} = 0,\\
&C_{-1,\phi\beta} = 0,\\
&C_{-1,\beta\phi} = 0,\\
&C_{-1,\beta\beta} = \bar{C}_{y\mathrm{f}}\cos^2\lambda + 2f\bar{C}_{y\mathrm{f}}\cos\lambda + f^2(\bar{C}_{y\mathrm{r}} + \bar{C}_{y\mathrm{f}}).
\end{aligned}
\qquad (69)
$$

The entries in the stiffness matrices are

$$\begin{aligned}
K_{0,\phi\phi} &= -f_\rho S_x g_x + S'_x g_z, \\
K_{0,\phi\beta} &= f_\mathrm{m} S_x g_x - (f_\mathrm{m} S'_z + S_\lambda) g_z, \\
K_{0,\beta\phi} &= [(f_\mathrm{m} - f)S_x - f m_\mathrm{T}\rho_\mathrm{r}]g_x - (f_\mathrm{m} S'_z + S_\lambda)g_z + M_\mathrm{f}[\rho_\mathrm{f}(\cos\lambda + f) - f\rho_\mathrm{r}]/r_\mathrm{f} \\
&\quad - f\bar{C}_{y\mathrm{r}}/r_\mathrm{r} - \bar{C}_{y\mathrm{f}}(\cos\lambda + f)/r_\mathrm{f} + f_\phi \bar{C}_{y\mathrm{f}}\cos\lambda + f_\phi f(\bar{C}_{y\mathrm{r}} + \bar{C}_{y\mathrm{f}}), \\
K_{0,\beta\beta} &= [f_\mathrm{m} S_x \sin\lambda + S_\lambda(\cos\lambda + f)]g_x - (f_\mathrm{m} S'_z + S_\lambda)\sin\lambda g_z \\
&\quad + M_\mathrm{f}[\rho_\mathrm{f}\sin\lambda(\cos\lambda + f) - f(t_\mathrm{pr} + w + t)\cos\lambda]/r_\mathrm{f} \\
&\quad - \bar{C}_{y\mathrm{f}}\sin\lambda\cos\lambda/r_\mathrm{f} - f\bar{C}_{y\mathrm{f}}\sin\lambda/r_\mathrm{f} + f_\beta \bar{C}_{y\mathrm{f}}\cos\lambda + f_\beta f(\bar{C}_{y\mathrm{r}} + \bar{C}_{y\mathrm{f}});
\end{aligned} \tag{70}$$

$$\begin{aligned}
K_{1,\phi\phi} &= (f_\rho + t_\mathrm{pr}/r_\mathrm{r})S_x - f_\rho S_\mathrm{w} + f_\phi I_{\mathrm{T}xz}, \\
K_{1,\phi\beta} &= -f_\mathrm{m}(S_x - S_\mathrm{w}) + S_\mathrm{f}\cos\lambda + f_\beta I_{\mathrm{T}xz}, \\
K_{1,\beta\phi} &= (f - f_\mathrm{m})(S_x - S_\mathrm{w}) + f m_\mathrm{T}\rho_\mathrm{r} - f S_z t_\mathrm{pr}/r_\mathrm{r} \\
&\quad - S_\lambda t_\mathrm{pr}/r_\mathrm{r} - S_\mathrm{f}\rho_\mathrm{f}(\cos\lambda + f)/r_\mathrm{f} + f\rho_\mathrm{r} S_\mathrm{f}/r_\mathrm{f} + f_\phi(I_{\mathrm{F}z\lambda} + f I_{\mathrm{T}zz}), \\
K_{1,\beta\beta} &= -f_\mathrm{m}(S_x - S_\mathrm{w})\sin\lambda - S_\lambda(\cos\lambda + f) \\
&\quad - S_\mathrm{f}[\rho_\mathrm{f}\sin\lambda(\cos\lambda + f) + f r_\mathrm{f}\sin\lambda - f(t_\mathrm{pr} + w + t)\cos\lambda]/r_\mathrm{f} + f_\beta(I_{\mathrm{F}z\lambda} + f I_{\mathrm{T}zz}),
\end{aligned} \tag{71}$$

$$\begin{aligned}
K_{2,\phi\phi} &= \bar{C}_\mathrm{d} z_\mathrm{d}(f_\rho + t_\mathrm{pr}/r_\mathrm{r}) - f_\phi(S_x - S_\mathrm{w}) - f_\phi \bar{C}_\mathrm{d} z_\mathrm{d}(t_\mathrm{pr} + x_\mathrm{d}), \\
K_{2,\phi\beta} &= -\bar{C}_\mathrm{d} z_\mathrm{d} f_\mathrm{m} - f_\beta(S_x - S_\mathrm{w}) - f_\beta \bar{C}_\mathrm{d} z_\mathrm{d}(t_\mathrm{pr} + x_\mathrm{d}), \\
K_{2,\beta\phi} &= -\bar{C}_\mathrm{d} z_\mathrm{d} f_\mathrm{m} + f\bar{C}_\mathrm{d}[\rho_\mathrm{r} + z_\mathrm{d} - (t_\mathrm{pr} + x_\mathrm{d})t_\mathrm{pr}/r_\mathrm{r}] \\
&\quad + f_\phi S_\lambda + f_\phi S_\mathrm{f}\sin\lambda + f_\phi f S_z + f_\phi f \bar{C}_\mathrm{d}(t_\mathrm{pr} + x_\mathrm{d})^2, \\
K_{2,\beta\beta} &= -\bar{C}_\mathrm{d} z_\mathrm{d} f_\mathrm{m}\sin\lambda + f_\beta S_\lambda + f_\beta S_\mathrm{f}\sin\lambda + f_\beta f S_z + f_\beta f \bar{C}_\mathrm{d}(t_\mathrm{pr} + x_\mathrm{d})^2 \,.
\end{aligned} \tag{72}$$

The coefficients of $\mathbf{K}^\mathrm{k}$ are

$$\begin{aligned}
K^\mathrm{k}_{\phi\psi} &= -S_x g_x, \\
K^\mathrm{k}_{\beta\psi} &= (S_\lambda + f S_z) g_x \,.
\end{aligned} \tag{73}$$

The kinematic differential equation (34) for the yaw angle ψ must be added for a complete specification of the system.

4 COMPARISON BETWEEN ANALYTIC AND NUMERIC EQUATIONS

The equations obtained in the previous section were compared with the results obtained with the multibody dynamic program SPACAR, which allows models that contain non-holonomic constraints and provides an option for generating linearized equations [10]. A special element for modelling the wheels with toroidal tread shape was developed along lines similar to the element for the wheel–rail contact in railway vehicles [11].

The matrices for the linearized equations for the example bicycle with parameters as in Table 1, an inclination angle of $5°$ and a braking moment of $M_\mathrm{f} = -35$ Nm on the front wheel are as follows:

$$\mathbf{M} = \begin{pmatrix} 80.817\,220\,000\,000\,00 & 2.752\,893\,706\,400\,66 \\ 2.752\,893\,706\,400\,66 & 0.343\,234\,252\,366\,12 \end{pmatrix}, \tag{74}$$

$$\mathbf{C}_1 = \begin{pmatrix} -3.967\,332\,330\,827\,07 & 35.629\,153\,284\,218\,26 \\ -0.995\,448\,919\,318\,55 & 1.992\,731\,670\,056\,25 \end{pmatrix}, \tag{75}$$

$$\mathbf{C}_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0.237\,873\,392\,539\,10 \end{pmatrix}, \tag{76}$$

$$\mathbf{K}_0 = \begin{pmatrix} -774.604\,923\,530\,537 & -28.824\,163\,496\,591 \\ -25.305\,268\,525\,705 & -0.071\,244\,904\,988 \end{pmatrix}, \tag{77}$$

$$\mathbf{K}_1 = \begin{pmatrix} -3.692\,636\,252\,395\,69 & 34.372\,172\,084\,873\,90 \\ -1.260\,555\,771\,598\,77 & 3.474\,695\,170\,872\,98 \end{pmatrix}, \tag{78}$$

$$\mathbf{K}_2 = \begin{pmatrix} 2.051\,757\,747\,309\,45 & 75.373\,607\,778\,119\,36 \\ 0.081\,128\,081\,694\,05 & 3.062\,902\,668\,239\,59 \end{pmatrix}, \tag{79}$$

$$\mathbf{K}^{\mathrm{k}} = \begin{pmatrix} 69.212\,074\,852\,898\,92 \\ 2.639\,816\,554\,532\,66 \end{pmatrix}, \tag{80}$$

$$\begin{aligned} f_\phi &= 0.025\,062\,656\,641\,60, \\ f_\beta &= 0.916\,629\,286\,468\,41, \\ f &= 0.085\,279\,921\,539\,14\,. \end{aligned} \tag{81}$$

These results were confirmed by the numerical analysis, barring some differences in the last few digits. Also in some other cases, the results agreed within numerical errors. This gives us some confidence that both the analytic derivation of the equations and the implementation in the program are correct.

5 DISCUSSION

Some of the implications of the modifications proposed in this paper on the equations will now be discussed. The mass matrix hardly changes: the increased value of f due to the pneumatic trail gives a slightly stronger coupling between the lean angle and the steering angle. The part of the damping matrix that is proportional to the forward speed is influenced by the aerodynamic drag, which gives a damping on the lean angle. The spin damping at the contact points gives a damping term that is inversely proportional to the forward speed; for very low speeds, this damping dominates one entry in the damping matrix and the normal spin at the front wheel is effectively suppressed. It should be noted, however, that this damping term becomes inaccurate at low speeds and invalid at zero speed. The pneumatic trails give some small further coupling terms to the mass distribution.

The finite transverse tire radius only modifies the constant part of the stiffness matrix, and indirectly, through the drag, the part that is proportional to the square of the velocity. Because the contact point shifts in lateral direction if the bicycle rolls, the capsize instability is reduced in strength.

The pneumatic trail influences some coupling terms. The most important influence is the increase of the factor f due to the pneumatic trail of the front wheel, which is especially significant if the mechanical trail is small.

The longitudinal forces that contribute to the acceleration of the bicycle have a contribution that is common to that of a driving torque at the rear wheel and which is described by the matrix $\mathbf{K}_1$, and some additional influences. This shows that the way in which the bicycle is accelerated or decelerated has an influence on the lateral dynamics. In particular, driving the bicycle at the rear wheel an simultaneously braking at the front wheel to keep the speed constant can improve the stability.

If there is a gradient, the yaw angle is no longer a cyclic coordinate and the neutral stability is lost: the bicycle has either a weak directional stability or a weak directional instability. In most cases, it has a tendency to steer towards the downhill direction, which means that riding down a slope leads to directional stability and riding up a slope leads to directional instability. Effects on the other eigenmodes are generally more important, however.

6 CONCLUSIONS

The linearized equations for the lateral motion of a bicycle have been extended to include the effects of the finite transverse radius of the tires, the pneumatic trail and the spin damping at the tire contact patches, driving and braking of the bicycle, aerodynamic drag and riding on an incline. The analytically derived equations have been compared with results from a multibody dynamic program and a satisfactory agreement has been found. This is a strong indication that the analytic derivation is correct and the toroidal wheels have been correctly implemented in the program at least in the linear range. The model can be used for further investigations into the dynamics and control of bicycles and be further extended to include non-linear effects.

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