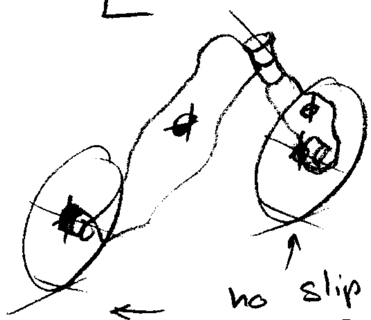
# Notes on the bicycle Project
## TAM 674



no slip!
in 2 Directions

| | | | | |
|---|---|---|---|---|
| 4 Bodies | -> | $4 \times 6$ | = 24 | coord |
| 3 Hinges | -> | $3 \times 5$ | = 15 | constraints |
| 2 Contact Points | -> | $2 \times 1$ | = 2 | constraints on coord |
| | | $2 \times 2$ | 4 | constraints on velocities |

24 coordinates
17 constraints on coordinates
+ 4 constraints on the velocities

Leaves $(24-21) = 3$ Degrees of Freedom (in the velocities)

$24 - 17 = 7$ Independent Coordinates

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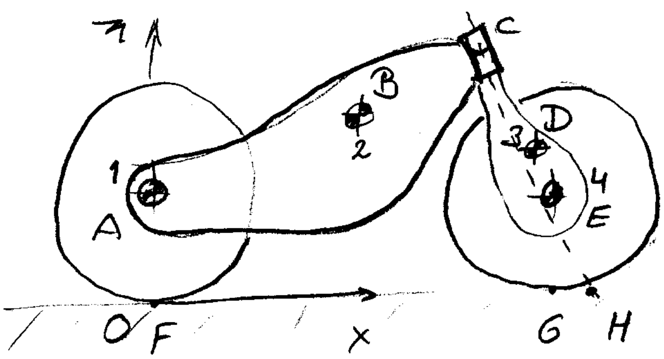
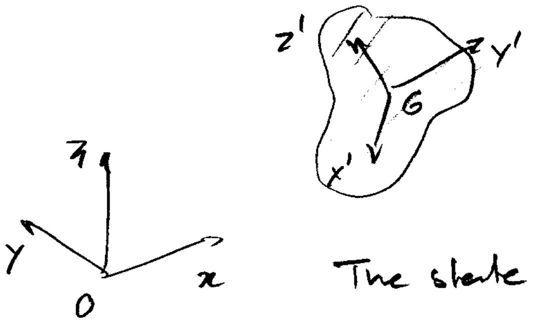
Look at the bodies and number them.




Pick a space fixed coord system O-xyz

Label a number of interesting points A, B, C ...

Equations of Motion for a single unconstrained Rigid Body




$\Sigma \underline{f} = m \, \dot{\underline{v}}_G$, Newton

$\Sigma \underline{M}'_G = \underline{J}'_G \, \dot{\underline{\omega}}' + \underline{\omega}' \times ( \underline{J}'_G \underline{\omega})$, Euler

The state is described by

$\underline{x}_G$, $\underline{p}$ being Euler Angles or Cardan Angles or Euler parameters or ....

$\underline{v}_G$, $\underline{\omega}'$

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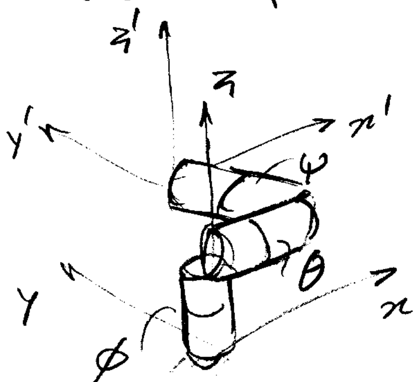
For the description of the orientation we will use the follow "Euler Angles" $\underline{p} = (\phi, \theta, \varphi)$ as depicted by:




$\phi$ about z-axis: Heading, Steer or Yaw

$\theta$ about rotated x-axis: Bank or Lean

$\varphi$ about rotated y-axis: Pitch

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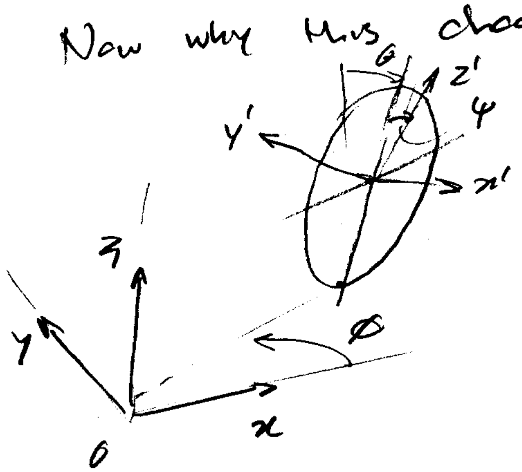
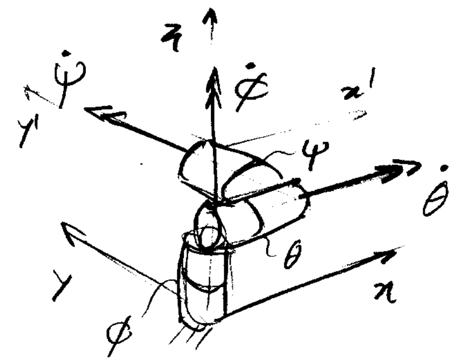
Now why this choose? Look at a wheel:




The last rotation, the Pitch $\varphi$, is the big one and therefore the last.

The singular configuration is at a Bank angle of $\theta = \pm 90^\circ$ which we try to avoid in cycling.

First, derive the rotation matrices for this construction and the angular velocities




$$\underline{x} = R_\phi R_\theta R_\psi \underline{x}'$$

with $R_\phi = \begin{pmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{pmatrix}$ $R_\psi = \begin{pmatrix} c_\psi & 0 & s_\psi \\ 0 & 1 & 0 \\ -s_\psi & 0 & c_\psi \end{pmatrix}$

Next the angular velocities:

$\underline{\omega} = A(\underline{p}) \cdot \dot{\underline{p}}$

$$\underline{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix} + R_\phi \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix} + R_\phi R_\theta \begin{pmatrix} 0 \\ \dot\psi \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}_{\underline{\omega}} = \underbrace{\begin{pmatrix} 0 & c_\phi & -s_\phi c_\theta \\ 0 & s_\phi & c_\phi c_\theta \\ 1 & 0 & s_\theta \end{pmatrix}}_{A(\underline{p})} \underbrace{\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix}}_{\dot{\underline{p}}}$$

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$\underline{\omega}' = A'(\underline{p})\, \dot{\underline{p}}$

$$\underline{\omega}' = \begin{pmatrix} 0 \\ \dot\psi \\ 0 \end{pmatrix} + R_\psi^T \begin{pmatrix} \dot\theta \\ 0 \\ 0 \end{pmatrix} + R_\psi^T R_\theta^T \begin{pmatrix} 0 \\ 0 \\ \dot\phi \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix}}_{\underline{\omega}'} = \underbrace{\begin{pmatrix} -s_\psi c_\theta & c_\psi & 0 \\ s_\theta & 0 & 1 \\ c_\psi c_\theta & s_\psi & 0 \end{pmatrix}}_{A'(\underline{p})} \underbrace{\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix}}_{\dot{\underline{p}}}$$

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And the Inverse:

$$\dot{\underline{p}} = (A'(\underline{p}))^{-1} \cdot \underline{\omega}'$$

$$\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} = \frac{1}{c_\theta} \begin{pmatrix} -s_\psi & 0 & c_\psi \\ c_\theta c_\psi & 0 & c_\theta s_\psi \\ s_\theta s_\psi & c_\theta & -s_\theta c_\psi \end{pmatrix} \begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix}$$

Singularity at $\theta = \pi/2 \pm k\pi$ !

# The State Equations:



$$
\left.
\begin{aligned}
\dot{\underline{x}}_{G_i} &= \underline{v}_{G_i} \\
\dot{\underline{p}}_i &= \left( A'(p_i) \right)^{-1} \underline{\omega}'_i
\end{aligned}
\right\} \text{ for each body } i=1 \ldots 4
$$

Constrained Equations of motion:

$$
\left[
\begin{array}{cccc:c}
m_1 I_3 & & & & \\
& J'_{G_1} & & & \\
& & m_2 I_3 & & \mathbb{D}^T \\
& & & J'_{G_2} & \\
& & & \ddots & \\
\hdashline
& & \mathbb{D} & & \emptyset
\end{array}
\right]
\left[
\begin{array}{c}
\dot{\underline{v}}_{G_1} \\
\dot{\underline{\omega}}'_1 \\
\dot{\underline{v}}_{G_2} \\
\dot{\underline{\omega}}'_2 \\
\vdots \\
\hdashline
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_{21}
\end{array}
\right]
=
\left[
\begin{array}{c}
\Sigma \underline{f}_1 \\
\Sigma \underline{M}'_1 - \underline{\omega}'_1 \times (J'_{G_1} \underline{\omega}'_1) \\
\Sigma \underline{f}_2 \\
\Sigma \underline{M}'_2 - \underline{\omega}'_2 \times \ldots \\
\vdots \\
\hdashline
\text{“} - \mathbb{D}^2 (\underline{v}_{G_i}, \underline{\omega}'_i) * (\underline{v}_{G_i}, \underline{\omega}'_i) \text{”}
\end{array}
\right]
$$

$g_i$ convective terms

where this $\mathbb{D}$ is the $\mathbb{D}_{h,j}$, the Jacobian of the constraints $\varepsilon_h = \mathbb{D}_h(\underline{x}_{G_j}, \underline{p}_j)$

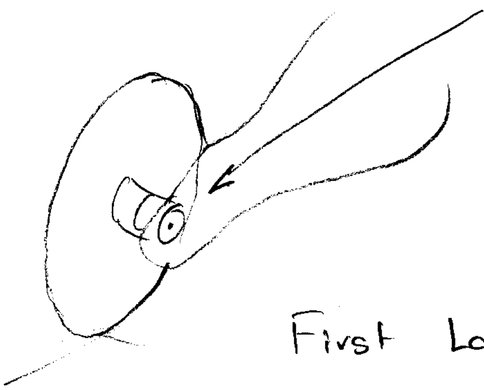
and the "Jacobian" of the velocity constraints

$$\dot{\varepsilon}_{nh} = \mathbb{D}_{nh} * (\dot{\underline{x}}_{G_j}, \dot{\underline{p}}_j) \; ?$$

wait and see ....!

## Constraints





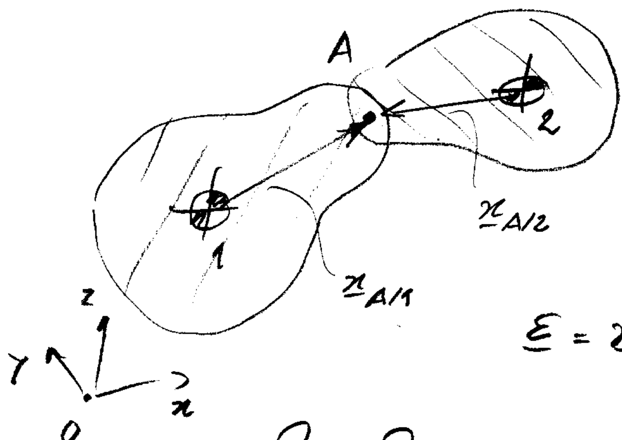
Revolute Joint =

2 Points coincide $\rightarrow$ 3

2 hinge axes parallel $\rightarrow$ 2

$\overline{\phantom{00}}$ 5

First Look at two Points Coincide:




$$\underline{\varepsilon} = \underline{x}_{A2} - \underline{x}_{A1} = \begin{pmatrix} \varepsilon_X \\ \varepsilon_Y \\ \varepsilon_Z \end{pmatrix}$$

$$\underline{\varepsilon} = \underline{x}_2 + \underline{x}_{A/2} - \underline{x}_1 - \underline{x}_{A/1}$$

$$\underline{\varepsilon} = \underline{x}_2 + R_2\, \underline{x}'_{A/2} - \underline{x}_1 - R_1\, \underline{x}'_{A/1}$$

constant vector!

$R_2 = R(\beta_2)$
$R_1 = R(\beta_1)$

Velocities:
$$\dot{\underline{\varepsilon}} = \dot{\underline{x}}_2 + \underline{\omega}_2 \times \underline{x}_{A/2} - \dot{\underline{x}}_1 - \underline{\omega}_1 \times \underline{x}_{A/1}$$
$$= \underline{V}_2 + \underline{\omega}_2 \times \underline{x}_{A/2} - \underline{V}_1 - \underline{\omega}_1 \times \underline{x}_{A/1}$$

Accelerations!
$$\ddot{\underline{\varepsilon}} = \dot{\underline{V}}_2 + \dot{\underline{\omega}}_2 \times \underline{x}_{A/2} - \dot{\underline{V}}_1 - \dot{\underline{\omega}}_1 \times \underline{x}_{A/1}$$
$$+ \underline{\omega}_2 \times \underline{\omega}_2 \times \underline{x}_{A/2} - \underline{\omega}_1 \times \underline{\omega}_1 \times \underline{x}_{A/1}$$

These last equations are the constraint eqn's we have to add to the eqn's of motion but...

we work in body fixed $\underline{\omega}'$ instead of space fixed $\underline{\omega}$

Velocities: $\dot{\underline{\varepsilon}} = \underline{v}_2 + R_2(\underline{\omega}_2' \times \underline{x}_{A2}') - \underline{v}_1 - R_1(\underline{\omega}_1' \times \underline{x}_{A1}')$

Which we can rewrite in a matrix vector expression by using the tilde matrix for the × product

$$\dot{\underline{\varepsilon}} = \underline{v}_2 - R_2 \underline{\tilde{x}}_{A2}' \underline{\omega}_2' - \underline{v}_1 + R_1 \underline{\tilde{x}}_{A1}' \underline{\omega}_1'$$

or in matrix-vector form

$$\dot{\underline{\varepsilon}} = \begin{bmatrix} -I & R_1 \underline{\tilde{x}}_{A1}' & I & -R_2 \underline{\tilde{x}}_{A2}' \end{bmatrix} \begin{bmatrix} \underline{v}_1 \\ \underline{\omega}_1' \\ \underline{v}_2 \\ \underline{\omega}_2' \end{bmatrix}$$

This matrix is the Jacobian of the constraints, "D"

The constraint on the accelerations is now

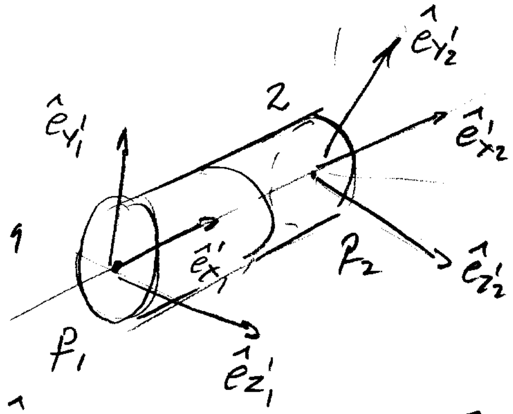
$$\ddot{\underline{\varepsilon}} = \begin{bmatrix} -I & R_1 \underline{\tilde{x}}_{A1}' & I & -R_2 \underline{\tilde{x}}_{A2}' \end{bmatrix} \begin{bmatrix} \dot{\underline{v}}_1 \\ \dot{\underline{\omega}}_1' \\ \dot{\underline{v}}_2 \\ \dot{\underline{\omega}}_2' \end{bmatrix} + \underline{g}$$

convective terms

were the convective terms $\underline{g}$ can be calculated from the state as

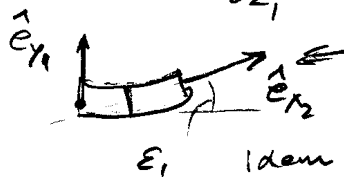
$$\underline{g} = \underline{\omega}_2 \times \underline{\omega}_2 \times \underline{\tilde{x}}_{A2}' - \underline{\omega}_1 \times \underline{\omega}_1 \times \underline{\tilde{x}}_{A1}$$

Next look at 2 Hinge axis being parallel constraint




We want the two hinge axis $\hat{e}_{x_1}$ and $\hat{e}_{x_2}$ to be parallel or in other words

$$\hat{e}_{x_2} \perp \hat{e}_{y_1} \quad \text{and} \quad \hat{e}_{x_2} \perp \hat{e}_{z_1}$$




$$\varepsilon_1 = \hat{e}_{y_1}^T \cdot \hat{e}_{x_2} \qquad (\text{like bending about } \hat{e}_{z'_1})$$

$$\varepsilon_2 = \hat{e}_{z_1}^T \cdot \hat{e}_{x_2} \qquad (\text{like bending about } \hat{e}_{y'_1})$$

or with $R_1 = R(P_1)$ and $R_2 = R(P_2)$ and referring to the body fixed prime vectors which are constant:

$$\varepsilon_1 = (R_1 \hat{e}_{y'_1})^T \cdot (R_2 \hat{e}_{x'_2}) = \hat{e}_{y'_1}^T \cdot R_1^T R_2 \hat{e}_{x'_2}$$

$$\varepsilon_2 = (R_1 \hat{e}_{z'_1})^T \cdot (R_2 \hat{e}_{x'_2}) = \hat{e}_{z'_1}^T \cdot R_1^T R_2 \hat{e}_{x'_2}$$

Velocities:

$$\dot{\varepsilon}_1 = \dot{\hat{e}}_{y_1}^T \cdot \hat{e}_{x_2} + \hat{e}_{y_1}^T \cdot \dot{\hat{e}}_{x_2}$$

$$= (\underline{\omega}_1 \times \hat{e}_{y_1})^T \hat{e}_{x_2} + \hat{e}_{y_1}^T \cdot (\underline{\omega}_2 \times \hat{e}_{x_2})$$

But we prefer the body fixed coord. syst $\underline{\omega}'$

$$\dot{\varepsilon}_1 = \left( R_1 (\underline{\omega}'_1 \times \hat{e}_{y'_1}) \right)^T \hat{e}_{x_2} + \hat{e}_{y_1}^T \left( R_2 (\underline{\omega}'_2 \times \hat{e}_{x'_2}) \right)$$

$$\dot{\varepsilon}_1 = - \hat{e}_{x_2}^T \cdot R_1 \tilde{\hat{e}}_{y'_1} \underline{\omega}'_1 - \hat{e}_{y_1}^T \cdot R_2 \tilde{\hat{e}}_{x'_2} \underline{\omega}'_2$$

Apparently the Jacobian "D" is

$$\frac{\partial \dot{\varepsilon}_1}{\partial \underline{\omega}'_1} = - \hat{e}_{x_2}^T \cdot R_2 \tilde{\hat{e}}_{y'_1} \quad \text{and} \quad \frac{\partial \dot{\varepsilon}_1}{\partial \underline{\omega}'_2} = - \hat{e}_{y_1}^T R_2 \tilde{\hat{e}}_{x'_2}$$

Accelerations:

$$\ddot{\varepsilon}_1 = (\dot{\underline{\omega}}_1 \times \hat{e}_{y1})^T \hat{e}_{x2} + \hat{e}_{y1}^T (\dot{\underline{\omega}}_2 \times \hat{e}_{x2}) +$$

$$\left\{ \begin{aligned} &(\underline{\omega}_1 \times \underline{\omega}_1 \times \hat{e}_{y1})^T \hat{e}_{x2} + (\underline{\omega}_1 \times \hat{e}_{y1})^T (\underline{\omega}_2 \times \hat{e}_{x2}) + \\ &(\underline{\omega}_1 \times \hat{e}_{y1})^T (\underline{\omega}_2 \times \hat{e}_{x2}) + \hat{e}_{y1}^T (\underline{\omega}_2 \times \underline{\omega}_2 \times \hat{e}_{x2}) \end{aligned} \right.$$

$= g_1$ the convective term which go to the right-hand side of the Eqns of Motion

and of course $\dfrac{\partial \ddot{\varepsilon}_1}{\partial \dot{\underline{\omega}}_1} = \dfrac{\partial \dot{\varepsilon}_1}{\partial \underline{\omega}_1}$ so this Jacobian is the same as the one from the velocities, allways!

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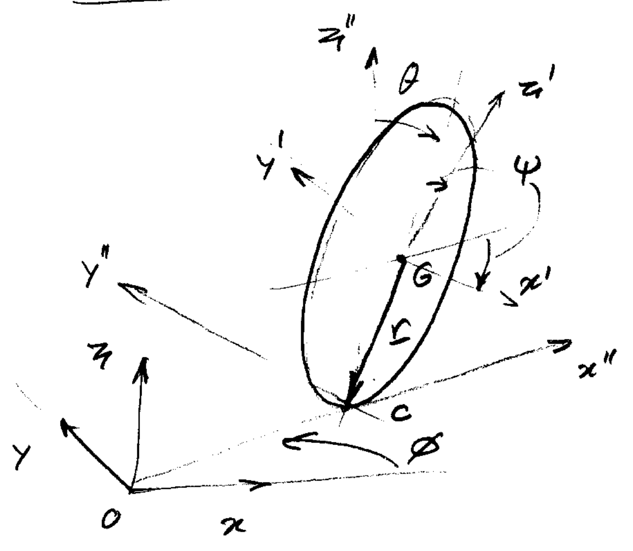
Wheel Contact Constraints:




Contact point c

$\dot{\underline{x}}_c = \dot{\underline{x}}_G + \underline{\omega} \times \underline{r}$

No slip means $\dot{\underline{x}}_c = \underline{0}$

Actually we 1 contact condition

$z_G = r \cdot \cos\theta$

and two no-slip conditions

$\dot{x}_c = 0$

$\dot{y}_c = 0$

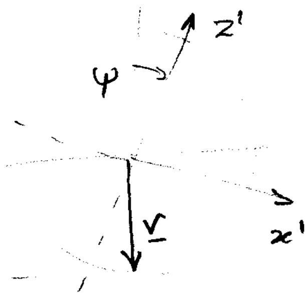
But .... I like to think in a moving coörd system 

$\dot{x}_c'' \neq 0$ Longitudinal slip

$\dot{y}_c'' \neq 0$ Lateral slip

$$\dot{\underline{x}}_c'' = \dot{\underline{x}}_G'' + \underline{\omega}'' \times \underline{v}''$$

Now $\dot{\underline{x}}_G'' = R_\phi^T \, \dot{\underline{x}}_G$ and $(\underline{\omega}'' \times \underline{v}'') = R_\theta R_\varphi (\underline{\omega}' \times \underline{v}')$




$$\Rightarrow \quad \underline{r}' = \begin{pmatrix} r S\varphi \\ 0 \\ -r C\varphi \end{pmatrix}$$

After evaluating $\dot{\underline{x}}_c''$ (use Matlab symbolic!!)

$$\left\{\begin{bmatrix} \dot{x}'' \\ \dot{y}'' \\ \dot{z}'' \end{bmatrix}\right\} = \begin{array}{c} \begin{array}{cccccc} \dot{x} & \dot{y} & \dot{z} & \omega_x' & \omega_y' & \omega_z' \end{array} \\ \begin{bmatrix} C\phi & S\phi & 0 & 0 & -r & 0 \\ -S\phi & C\phi & 0 & rC_\theta C\varphi & 0 & rC_\theta S\varphi \\ 0 & 0 & 1 & rS_\theta C\varphi & 0 & rS_\theta S\varphi \end{bmatrix} \end{array}$$

This last one is the constraint $\dot{z} + r\sin\theta\,\dot{\theta} = 0$

(check this with $\dot{\theta}$ from $\dot{\underline{p}} = (A'(\underline{p}))^{-1}\underline{\omega}'$ in terms of $\underline{\omega}'$ i.e. $\dot{\theta} = C\varphi\,\omega_x' + S\varphi\,\omega_z'$ )

which comes from the holonomic contact condition

$$\mathcal{E}_1 = Z - r\cos\theta$$

The first two are the two non-holonomic constraints

$$\dot{\mathcal{E}}_2 = C\phi\,\dot{x} + S\phi\,\dot{y} - r\,\omega_y' \qquad \text{(Longitudinal Slip)}$$

$$\dot{\mathcal{E}}_3 = -S\phi\,\dot{x} + C\phi\,\dot{y} + r\omega_x' C_\theta C\varphi + r\omega_z' C_\theta S\varphi \qquad \text{(Lateral Slip)}$$

Now we only have to find the convective terms for the wheel constraint. First substitute $\underline{\omega}$ in terms of $\underline{p}$ and $\underline{\dot{p}}'$ in the $\underline{\dot{\varepsilon}}$ eqn's

$$\begin{array}{c} \\ \begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \end{bmatrix} = \end{array} \begin{array}{c} [\ \dot{x} \quad \dot{y} \quad \dot{z} \quad \dot{\phi} \quad \dot{\theta} \quad \dot{\psi}\ ] \\ \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & rS_\theta & 0 \\ C_\phi & S_\phi & 0 & -rS_\theta & 0 & -r \\ -S_\phi & C_\phi & 0 & 0 & rC_\theta & 0 \end{array}\right] \end{array}$$

Next diff with respect to time and forget all $\ddot{\ }$ gives us the convective terms

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} rC_\theta \dot{\theta}^2 \\ S_\phi \dot{x}\dot{\phi} + C_\phi \dot{y}\dot{\phi} - rC_\theta \dot{\phi}\dot{\theta} \\ -C_\phi \dot{x}\dot{\phi} - S_\phi \dot{y}\dot{\phi} - rS_\theta \dot{\theta}^2 \end{bmatrix}$$

And finally substitute $\underline{\dot{p}} = (A'(p))^{-1}\underline{\omega}'$ to get $\underline{g}$ in terms of the state variables $\underline{p}$, $\underline{v}$ and $\underline{\omega}'$

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Use the "coordinate projection method" to minimize the constraint errors after one numerical integration step. In this scheme we use the Moore-Penrose pseudo-inverse $D^+$ the Jacobian of the constraints $D$ as in $D^+ = D^T(DD^T)^{-1}$.

Where the Jacobian $D = \dfrac{\partial \underline{\varepsilon}}{\partial(\underline{x}_i, \underline{p}_i)}$, the partial derivatives of the constraints with the respect to the state variables $\underline{x}_i$ and $\underline{p}_i$. This is not equal to the $D$ which we have used in the constraint eqn's of motion, since there we had the partial derivatives of the velocities with respect to the state variable $\underline{v}_i$ and $\underline{\omega}'_i$, as in $D = \dfrac{\partial \dot{\underline{\varepsilon}}}{\partial(\underline{v}_i, \underline{\omega}'_i)}$.

But given the latter we can easily transform to the former with the matrices $(A'(\beta_i))^{-1}$ since

$\dfrac{\partial \underline{\omega}'_i}{\partial \beta_i} = (A'(\beta_i))^{-1}$ and thus

$$\frac{\partial \underline{\varepsilon}}{\partial(\underline{x}_i, \beta_i)} = \frac{\partial \dot{\underline{\varepsilon}}}{\partial(\underline{v}_i, \underline{\omega}'_i)} \cdot (A'(\beta_i))^{-1}$$

As an example the <u>two Points Coincide</u> constraint for the eqn of we derived

$$\dot{\underline{\varepsilon}} = \begin{bmatrix} -I & R_1 \tilde{\underline{x}}'_{A/1} & I & -R_2 \tilde{\underline{x}}'_{A/2} \end{bmatrix} \begin{bmatrix} \underline{v}_1 \\ \underline{\omega}'_1 \\ \underline{v}_2 \\ \underline{\omega}'_2 \end{bmatrix}$$

which transform to

$$\Delta \underline{\varepsilon} = \begin{bmatrix} -I & R_1 \tilde{\underline{x}}'_{A/1} (A'_1)^{-1} & I & -R_2 \tilde{\underline{x}}'_{A/2} (A'_2)^{-1} \end{bmatrix} \begin{bmatrix} \Delta \underline{x}_1 \\ \Delta \beta_1 \\ \Delta \underline{x}_2 \\ \Delta \beta_2 \end{bmatrix}$$

Note that this "coordinate projection method" only operates on the holonomic constraints!

in our case 17 constraints in total

Finally we have to find speeds, $\underline{v}_i$ and $\underline{\omega}_i$ which fulfil the velocity constraints. So now we include our non-holonomic constraints. To minimize the velocity constant errors we use the "coordinate projection method" again but since we work in this velocity space we can use the Jacobian "D" as derived for the constrained eqn's of motion. Since the eqn's are linear in the speeds this is a one step iteration!

DONE

Arend L. Schwab

Ithaca, April 29, 2003