HIGHER MECHANICS

BY

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Lagrange's equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = \Phi \quad \text{.............................(19)}$$

accordingly becomes

$$C\frac{dr}{dt} - (A - B)pq = N. \quad \text{..........................(20)}$$

Since it is indifferent which of the principal axes at the fixed point we denote by OC, the remaining equations of Euler's triad will also hold.

Ex. 2. In the steam-engine the driving power is usually controlled by some form of 'centrifugal governor.' The original type, introduced by Watt, is shewn in the annexed sketch. The spindle to which the arms carrying the two balls are hinged rotates at a rate proportional to the speed of the engine. When this rotation is uniform the balls, under the action of gravity and centrifugal force, take up a definite 'equilibrium' position depending on the speed. If the speed increases the balls diverge outwards, raising the collar c to which the lower arms are connected, and thus operating a system of levers which turn a valve so as to reduce the supply of steam. Conversely, when the speed diminishes, the collar descends, and the supply is reinforced.

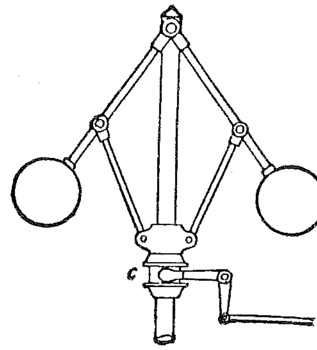

Fig. 56.

If θ denote the inclination of the upper arms to the spindle, and $\dot{\psi}$ the angular velocity about the vertical, the expression for the kinetic energy has the form

$$2T = A\dot{\theta}^2 + I\dot{\psi}^2, \quad \text{...............................(21)}$$

where A and I are functions of θ. The coefficient I is supposed to include a term representing the inertia of the engine and of the train of machinery in connection with it. Lagrange's formula gives

$$\frac{d}{dt}(A\dot{\theta}) - \frac{1}{2}\frac{\partial A}{\partial \theta}\dot{\theta}^2 - \frac{1}{2}\frac{\partial I}{\partial \theta}\dot{\psi}^2 = -\frac{\partial V}{\partial \theta}, \quad \text{....................(22)}$$

and

$$\frac{d}{dt}(I\dot{\psi}) = \Psi, \quad \text{..................................(23)}$$

where V is the potential energy of the governor, and Ψ represents the excess of driving power over resistance. If this excess vanishes when the valve has the position corresponding to $\theta = a$ we may write, as an approximation,

$$\Psi = -\beta(\theta - a). \quad \text{...............................(24)}$$

For steady motion we must have $\theta = a$, $\dot{\psi} = \omega$, where ω is determined by

$$\frac{1}{2}\frac{\partial I}{\partial \theta}\omega^2 = \frac{\partial V}{\partial \theta}. \quad \text{...............................(25)}$$

Since this involves the above value of θ, the speed ω will vary with any permanent change in the driving power. The contrivance does not therefore maintain a constant speed independent of variations in the driving power, and it was therefore suggested by Maxwell that it should properly be called a 'moderator' rather than a 'governor.'

To examine the effect of accidental disturbances of the steady motion, we write

$$\theta = a + x, \quad \dot{\psi} = \omega + y, \qquad \text{.........................(26)}$$

and treat x, y as small. If we cancel the terms which refer to the steady motion, the equations (22) and (23) become

$$A\ddot{x} - \tfrac{1}{2}I''\omega^2 x - I'\omega y + V''x = 0, \qquad \text{.....................(27)}$$

$$I\dot{y} + I'\omega\dot{x} + \beta x = 0, \qquad \text{..............................(28)}$$

where accents indicate differentiations with respect to θ, and the coefficients are supposed to have the constant values corresponding to $\theta = a$. Assuming that x and y vary as $e^{\lambda t}$, we find

$$AI\lambda^3 + \{I(V'' - \tfrac{1}{2}I'\omega^2) + I'^2\omega^2\}\lambda + I'\beta\omega = 0. \qquad \text{...............(29)}$$

It is essential of course that the governor should be stable when the speed ω is maintained constant. The condition for this is*

$$V'' - \tfrac{1}{2}I'\omega^2 > 0. \qquad \text{.................................(30)}$$

This being satisfied, the coefficients in (29) are all positive. There is therefore one negative, and no positive root. Since the sum of the roots is zero, the remaining roots must be imaginary with positive real part. The complete solution of (27) and (28) therefore consists of terms of the types $e^{-2\mu t}$, $e^{\mu t}\cos\nu t$, $e^{\mu t}\sin\nu t$. The latter pair indicate an oscillation of continually increasing amplitude.

This instability is checked to some extent by the inevitable friction between various parts of the mechanism, but in order definitely to eliminate it a viscous resistance is sometimes expressly introduced, opposing variations of θ. This may be represented by inserting a term $\gamma d\theta/dt$ on the right-hand side of (22), and therefore a term $\gamma\dot{x}$ in (27). The resulting equation in λ is

$$AI\lambda^3 + \gamma I\lambda^2 + \{I(V'' - \tfrac{1}{2}I'\omega^2) + I'^2\omega^2\}\lambda + I'\beta\omega = 0. \qquad \text{.........(31)}$$

There is obviously one negative root, as before. The condition that the remaining roots should be negative, or imaginary with negative real part, is†

$$\gamma I\{I(V'' - \tfrac{1}{2}I'\omega^2) + I'^2\omega^2\} > AII'\beta\omega, \qquad \text{..................(32)}$$

which is satisfied if the frictional coefficient γ is sufficiently great.

* In the problem of Art. 80, Ex. 1, we have

$$V = -mga\cos\theta, \quad I = ma^2\sin^2\theta,$$

and therefore $$V'' - \tfrac{1}{2}I'\omega^2 = m\omega^2 a\sin^2\theta,$$

in the position of relative equilibrium, where $\cos\theta = g/\omega^2 a$. A closer analogy to the circumstances of Watt's governor is furnished by Ex. 3 of Art. 80.

† If α, β, γ be the roots of the cubic

$$x^3 + p_1x^2 + p_2x + p_3 = 0,$$

we have $$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = p_3 - p_1p_2.$$