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HOW TO DRAW EULER ANGLES AND UTILIZE EULER PARAMETERS

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ABSTRACT

This article presents a way to draw Euler angles such that the proper operation and application becomes immediately clear. Furthermore, Euler parameters, which allow a singularity-free description of rotational motion, are discussed within the framework of quaternion algebra and are applied to the kinematics and dynamics of a rigid body.

1 Euler Angles

In rigid body mechanics we need to keep track of points for each body. The motion of such a body can be decomposed into a translation and a rotation. Here we focus on the rotational part. One way to describe the rotation (the change in orientation) of a rigid body is by means of Euler angles. Or more precisely: a way to parametrize the rotation matrix is to use the three Euler angles [1]. For a rotation about a fixed origin, the rotation matrix $\mathbf{R}$ is the orthogonal matrix which transforms the coordinates of a point r from the body fixed coordinate system to the space fixed coordinate system, as in

$$\mathbf{r} = \mathbf{R}\mathbf{r}', \tag{1}$$

with space fixed coordinates $\mathbf{r}$ and body fixed coordinates $\mathbf{r}'$. Since for a rigid body these body fixed coordinates are constant, Euler angles are a way to keep track of a point of the body in space.

So how do we find the coordinates of a point? We start with the position of a point r in the body given by the vector $\vec{\mathbf{r}}$. Of course, this vector stays the same, whatever coordinate system we use. Now let us assume that the space fixed coordinate system is spanned by the three orthogonal unit vectors $(\vec{\mathbf{e}}_x, \vec{\mathbf{e}}_y, \vec{\mathbf{e}}_z)$, also called the base vectors. To find the coordinates of the vector $\vec{\mathbf{r}}$ expressed in the space fixed coordinate system we write, $\vec{\mathbf{r}} = x\vec{\mathbf{e}}_x + y\vec{\mathbf{e}}_y + z\vec{\mathbf{e}}_z$. The coordinates are the three scalars x, y and z and a handy way to describe them is to group them in a list. This list is then called the coordinate vector $\mathbf{r} = (x, y, z)$. Note the difference: the vector is $\vec{\mathbf{r}}$ whereas the coordinates of this vector expressed in some coordinate system are $\mathbf{r}$. Eventually, if we want to make calculations, which means to get away from the formal description and to do actually something with numbers, it is $\mathbf{r}$, the coordinate vector, which we use. These are the numbers that go into the calculating program.

The distinction is not necessary if we use only one coordinate system, but in the case of a rotating rigid body we clearly identify two coordinate systems: a coordinate system glued on the body, which we call the body fixed coordinate system and denote by primed symbols $(\vec{\mathbf{e}}_{x'}, \vec{\mathbf{e}}_{y'}, \vec{\mathbf{e}}_{z'})$, and the space fixed coordinate system $(\vec{\mathbf{e}}_x, \vec{\mathbf{e}}_y, \vec{\mathbf{e}}_z)$ which is our reference system. Next we express the position of r in the body fixed coordinate system as in $\vec{\mathbf{r}} = x'\vec{\mathbf{e}}_{x'} + y'\vec{\mathbf{e}}_{y'} + z'\vec{\mathbf{e}}_{z'}$ and this defines the body fixed coordinates $\mathbf{r}' = (x', y', z')$ of point r. As already said, for a rigid body these are constant.

Now we can get back to the rotation matrix and the Euler angles. Most textbooks (e.g. Goldstein [1], Hamel [2], Wittenburg [3], Lurie [4], Papastavridis [5], Shabana [6]) introduce Euler angles (3–1–3) as follows. The body or body fixed coordinate system can be oriented with respect to the space fixed coordinate system by means of three successive rotations. The sequence starts by rotating the body fixed coordinate system, which is initially aligned with the space fixed coordinate system, by an angle ϕ about the $\vec{\mathbf{e}}_z$ axis. The resulting coordinate system is then labelled $(\vec{\mathbf{e}}_\xi, \vec{\mathbf{e}}_\eta, \vec{\mathbf{e}}_\zeta)$. In a second step the intermediate coordinate system $(\vec{\mathbf{e}}_\xi, \vec{\mathbf{e}}_\eta, \vec{\mathbf{e}}_\zeta)$ is rotated about the $\vec{\mathbf{e}}_\xi$ axis by an angle θ to produce yet another intermediate coordinate system denoted by $(\vec{\mathbf{e}}_{\xi'}, \vec{\mathbf{e}}_{\eta'}, \vec{\mathbf{e}}_{\zeta'})$. Finally this $(\vec{\mathbf{e}}_{\xi'}, \vec{\mathbf{e}}_{\eta'}, \vec{\mathbf{e}}_{\zeta'})$ coordinate system is rotated about the $\vec{\mathbf{e}}_{\zeta'}$ axis by an angle ψ to produce the body fixed coordinate system labelled $(\vec{\mathbf{e}}_{x'}, \vec{\mathbf{e}}_{y'}, \vec{\mathbf{e}}_{z'})$. The various stages of this sequence are then illustrated by figure 1.

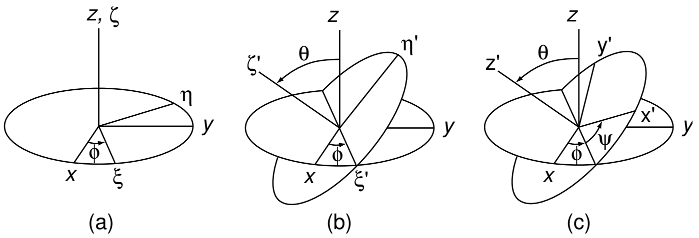

Figure 1. The three stages of rotation for Euler angles.

Most modern first-time readers are now totally lost. The process of successive rotation is complex and the drawing even more. Therefore we propose to illustrate this sequence of rotations about different axes by means of the so-called cans in series, figure 2. Each rotation about an axis is represented by a pair of cans rotating with respect to one another. Of course, the drawing is not entirely correct, the origins of the various coordinate systems do not coincide. This drawback is yet the power of the picture. The drawing of the cans in series can be looked upon as an exploded view of the materialization of the Euler angles and by such demonstrates the proper operation of the process. The

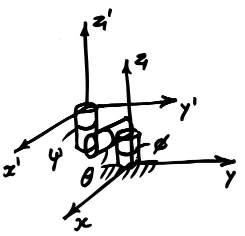

Figure 2. Euler angles as 'cans' in series.

non-coinciding origins are now immaterial and the role of the two intermediate coordinate systems becomes clearer. They are positioned at the the end of the first two pairs of cans.

The rotation matrix $\mathbf{R}$ is obtained by looking at the rotations of the individual pairs of cans, figure 3. The first pair of cans

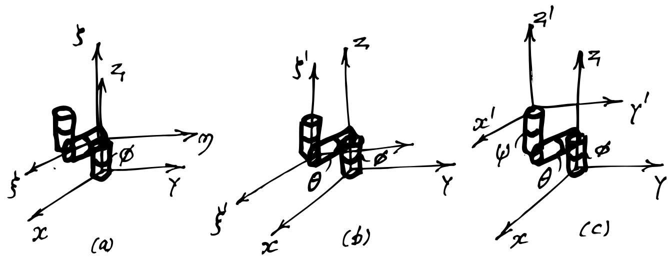

Figure 3. Euler angle sequence with 'cans' in series.

describe the rotation about the $\vec{\mathbf{e}}_z$ axis by an angle ϕ as in

$$\mathbf{r} = \mathbf{R}_\phi \rho, \quad \text{with} \quad \mathbf{R}_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

and the coordinates $\rho = (\xi, \eta, \zeta)$ of point r in the $(\vec{\mathbf{e}}_\xi, \vec{\mathbf{e}}_\eta, \vec{\mathbf{e}}_\zeta)$ coordinate system. The rotation matrix $\mathbf{R}_\phi$ has a simple form, because the rotation is about a coordinate axis. The second pair of cans describe the rotation about the $\vec{\mathbf{e}}_\xi$ axis by an angle θ:

$$\rho = \mathbf{R}_\theta \rho', \quad \text{with} \quad \mathbf{R}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{3}$$

and the coordinates $\rho' = (\xi', \eta', \zeta')$ of point r in the $(\vec{\mathbf{e}}_{\xi'}, \vec{\mathbf{e}}_{\eta'}, \vec{\mathbf{e}}_{\zeta'})$ coordinate system. Finally, the last pair of cans describe the rotation about the $\vec{\mathbf{e}}_{\zeta'}$ axis by an angle ψ:

$$\rho' = \mathbf{R}_\psi \mathbf{r}', \quad \text{with} \quad \mathbf{R}_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Substitution of (4) in (3) and (3) in (2) leads to the complete transformation of the body fixed coordinates to the space fixed coordinates, $\mathbf{r} = \mathbf{R}\mathbf{r}'$, where the rotation matrix $\mathbf{R}$ in terms of the three Euler angles (ϕ, θ, ψ) is the product of the three successive rotation matrices, as in

$$\mathbf{R} = \mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi. \tag{5}$$

Note the order in which the matrices are multiplied.

The inverse transformation of the space fixed coordinates to the body fixed coordinates

$$\mathbf{r}' = \mathbf{R}^{-1}\mathbf{r} \tag{6}$$

is then given immediately by its transpose

$$\mathbf{R}^{-1} = \mathbf{R}^T = \mathbf{R}_\psi^T\mathbf{R}_\theta^T\mathbf{R}_\phi^T \tag{7}$$

since $\mathbf{R}$ is an orthogonal matrix. This result can also be found by doing the successive rotations in reverse direction (angle $\to$ $-$angle) and in reverse order.

The expressions for the components of the angular velocities vector $\vec{\boldsymbol{\omega}}$ in terms of the Euler angles and their time derivatives are usually found by taking the time derivatives of (1), substitution of (6) and cancellation of $\dot{\mathbf{r}}'$, because the body fixed coordinates are constant, leading to

$$\dot{\mathbf{r}} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{r}. \tag{8}$$

The matrix $\dot{\mathbf{R}}\mathbf{R}^T$ is identified as an antisymmetric matrix because differentiation of the orthogonality conditions $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ leads to $\dot{\mathbf{R}}\mathbf{R}^T + (\dot{\mathbf{R}}\mathbf{R}^T)^T = \mathbf{O}$. This antisymmetric matrix is then called $\tilde{\boldsymbol{\omega}}$ and represents the cross product of the components $(\omega_x, \omega_y, \omega_z)$ of the angular velocity vector $\vec{\boldsymbol{\omega}}$ expressed in the body fixed coordinate system such that

$$\dot{\mathbf{r}} = \tilde{\boldsymbol{\omega}}\mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}. \tag{9}$$

Here we have used the tilde notation for the antisymmetric matrix $\tilde{\boldsymbol{\omega}}$ from the vector $\boldsymbol{\omega}$, which is defined by the matrix–vector notation for the vector cross product $\boldsymbol{\omega} \times \mathbf{x} = \tilde{\boldsymbol{\omega}}\mathbf{x}$. This antisymmetric matrix is

$$\tilde{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}. \tag{10}$$

The components of the angular velocity expressed in the space fixed coordinate system can now be found by equating the matrix $\tilde{\boldsymbol{\omega}}$ with the expanded partial derivatives from (8) as in

$$\tilde{\boldsymbol{\omega}} = \frac{\partial \mathbf{R}}{\partial \phi}\mathbf{R}^T\dot{\phi} + \frac{\partial \mathbf{R}}{\partial \theta}\mathbf{R}^T\dot{\theta} + \frac{\partial \mathbf{R}}{\partial \psi}\mathbf{R}^T\dot{\psi}. \tag{11}$$

This is a long an tedious road. A shortcut is given by inspection of figure 4. The rates of the Euler angles are drawn as angular velocity vectors at the corresponding pair of cans, with

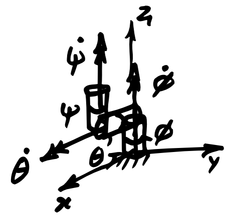

Figure 4. Euler angles and angular velocities.

$$\vec{\boldsymbol{\omega}}_\phi = \dot{\phi}\vec{\mathbf{e}}_z, \quad \vec{\boldsymbol{\omega}}_\theta = \dot{\theta}\vec{\mathbf{e}}_\xi, \quad \text{and} \quad \vec{\boldsymbol{\omega}}_\psi = \dot{\psi}\vec{\mathbf{e}}_{\zeta'}. \tag{12}$$

The angular velocity of the body is the sum of these successive angular velocities, $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}_\phi + \vec{\boldsymbol{\omega}}_\theta + \vec{\boldsymbol{\omega}}_\psi$. Then the components of the angular velocity expressed in the space fixed coordinate system $(\omega_x, \omega_y, \omega_z)$ are found by transforming the three angular velocity vectors to the space fixed coordinate system and adding them up, as in

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} + \mathbf{R}_\phi \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\phi\mathbf{R}_\theta \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \tag{13}$$

Next after expansion of terms we find

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 & \cos\phi & \sin\phi\sin\theta \\ 0 & \sin\phi & -\cos\phi\sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}, \quad \boldsymbol{\omega} = \mathbf{A}\dot{\mathbf{u}}, \tag{14}$$

where we have introduced the velocity transformation matrix $\mathbf{A}$ and the list of Euler angles $\mathbf{u} = (\phi, \theta, \psi)$. We say list, because it is not useful to consider them as a three-dimensional vector: the standard vector addition and multiplication by a scalar do not correspond to the composition of rotations. The components of the angular velocity of the body expressed in the body fixed coordinate system $\boldsymbol{\omega}' = (\omega'_x, \omega'_y, \omega'_z)$ can be found by transforming the angular velocities from (14) according to $\boldsymbol{\omega}' = \mathbf{R}^T\mathbf{A}\dot{\mathbf{u}}$. Another approach is to look at the series of cans from figure 4, and to transform the individual angular velocities from (12) to the body fixed coordinate system as in

$$\begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \mathbf{R}_\psi^T \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\psi^T\mathbf{R}_\theta^T \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} \tag{15}$$

which after expansion of terms gives us

$$\begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} \sin\psi\sin\theta & \cos\psi & 0 \\ \cos\psi\sin\theta & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix}, \quad \boldsymbol{\omega}' = \mathbf{B}\dot{\mathbf{u}}. \quad (16)$$

The velocity transformation matrices $\mathbf{A}$ and $\mathbf{B} = \mathbf{R}^T\mathbf{A}$ both have a determinant of $-\sin\theta$ and therefore show a singularity at $\theta = 0 \pm \pi$. In this configuration it is not possible to uniquely determine the rate of the Euler angles from the angular velocities. This is usually referred to as 'gimbal lock'. The gimbal lock becomes apparent if one looks at the pair of cans in the singular configuration, for example the initial configuration from figure 4. In this case all rotation axes of the cans are in one plane. No out-of-plane angular velocity, in this case ω_y, can be resolved by the rate of the Euler angles.

With the Euler angles as generalized coordinates we are able to derive the equations of motion of a rigid body in terms of Euler angles and their time derivatives. We start with the equations of motion for the rotation of a rigid body in space with the components of the inertia tensor as matrix $\mathbf{J}'$ and the vector of applied torques $\mathbf{M}'$, all at the centre of mass expressed in the body fixed frame, being

$$\mathbf{J}'\dot{\boldsymbol{\omega}}' = \mathbf{M}' - \boldsymbol{\omega}' \times (\mathbf{J}'\boldsymbol{\omega}'). \quad (17)$$

Next we apply the principle of virtual power

$$(\mathbf{M}' - \mathbf{J}'\dot{\boldsymbol{\omega}}' - \boldsymbol{\omega}' \times (\mathbf{J}'\boldsymbol{\omega}'))^T \delta\boldsymbol{\omega}' = 0 \quad \forall \quad \{\delta\boldsymbol{\omega}' = \mathbf{B}\delta\dot{\mathbf{u}}\}, \quad (18)$$

and substitute the angular velocities (16) and accelerations $\dot{\boldsymbol{\omega}}' = \mathbf{B}\ddot{\mathbf{u}} + \dot{\mathbf{B}}\dot{\mathbf{u}}$. The equations of motion for the rotation of a rigid body in terms of the Euler angles and their time derivatives are now

$$\mathbf{B}^T\mathbf{J}'\mathbf{B}\ddot{\mathbf{u}} = \mathbf{B}^T[\mathbf{M}' - (\mathbf{B}\dot{\mathbf{u}}) \times (\mathbf{J}'\mathbf{B}\dot{\mathbf{u}}) - \mathbf{J}'\dot{\mathbf{B}}\dot{\mathbf{u}}]. \quad (19)$$

Note that these equations show the same singularity at gimbal lock as the velocities (16).

A computationally far more efficient way to calculate the motion of a rigid body is not to transform the equations of motion to the generalized coordinates but instead to use the angular velocities $\boldsymbol{\omega}'$ together with the Euler angles $\mathbf{u}$ as state variables [8]. The state equations then become

$$\dot{\boldsymbol{\omega}}' = \mathbf{J}'^{-1}[\mathbf{M}' - \boldsymbol{\omega}' \times (\mathbf{J}'\boldsymbol{\omega}')], \quad (20)$$
$$\dot{\mathbf{u}} = \mathbf{B}^{-1}\boldsymbol{\omega}', \quad (21)$$

where the presence of gimbal lock becomes evident through the set of equations (21).

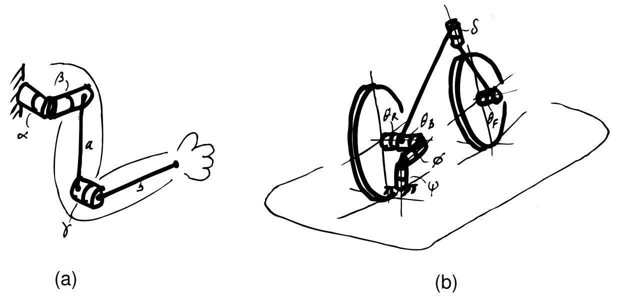

Figure 5. Two examples of the usage of cans in series for depicting rotational motion: (a) an arm-like manipulator, and (b) a bicycle model [9].

Finally, the pair of cans are successful not only for depicting Euler angles but also for illustrating relative rotation in general. In figure 5 on the left a model is shown of an arm-like manipulator, the cans drawn at the base show the proper direction and order of rotations α and β. The same figure on the right shows a bicycle model from [9] where the pair of cans at the rear hub are used in an Euler angle manner (ψ, ϕ, θ_B) but where the other pair of cans are used to illustrate the rear wheel rotation θ_R, the steering angle δ, and the front wheel rotation θ_F .

2 Quaternions, Finite Rotation, and Euler Parameters

The problem of gimbal lock can be resolved by using Euler parameters to parametrize the rotation matrix $\mathbf{R}$. Euler parameters are unit quaternions [7, 10]. A quaternion is a collection of four real parameters, of which the first is considered as a scalar and the other three as a vector in three-dimensional space. The following operations are defined. If $q = (q_0, \mathbf{q}) = (q_0, q_1, q_2, q_3)$ and $p = (p_0, \mathbf{p}) = (p_0, p_1, p_2, p_3)$ are two quaternions, their sum is defined as

$$q + p = (q_0 + p_0, \mathbf{q} + \mathbf{p}), \quad (22)$$

and their product (non-commutative) as

$$q \circ p = (q_0 p_0 - \mathbf{q}\cdot\mathbf{p}, q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{q} \times \mathbf{p}). \quad (23)$$

Although non-commutative, the quaternion product is associative and satisfies $r \circ (p \circ q) = (r \circ p) \circ q$. The adjoint quaternion of q is defined as $\overline{q} = (q_0, -\mathbf{q})$ and the length or norm as $|q| = \sqrt{(\overline{q} \circ q)_0} = \sqrt{q_0^2 + \mathbf{q}\cdot\mathbf{q}}$. Note that $|q \circ p| = |q||p|$. There are two special quaternions, the unit element $1 = (1, \mathbf{0})$ and the zero element $0 = (0, \mathbf{0})$. The reciprocal of a quaternion $q \neq 0$ is $q^{-1} = \overline{q}/|q|^2$. The quaternion with a norm of one, $|q| = 1$, is a unit quaternion.

If a quaternion is considered as a four-dimensional vector, the quaternion product can be described by a matrix–vector product as

$$q \circ p = \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & q_0\mathbf{I}_3 + \tilde{\mathbf{q}} \end{pmatrix} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \mathbf{Q} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix},$$
$$p \circ q = \begin{pmatrix} q_0 & -\mathbf{q}^T \\ \mathbf{q} & q_0\mathbf{I}_3 - \tilde{\mathbf{q}} \end{pmatrix} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix} = \overline{\mathbf{Q}} \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix}. \qquad (24)$$

Any pair of quaternion matrices $\mathbf{Q}$ and $\overline{\mathbf{P}}$ commute, $\mathbf{Q}\overline{\mathbf{P}} = \overline{\mathbf{P}}\mathbf{Q}$. The matrices of the adjoint quaternion $\overline{q}$ are $\mathbf{Q}^T$ and $\overline{\mathbf{Q}}^T$.

If we associate the quaternion $x' = (0, \mathbf{x}')$ with the three-dimensional vector $\mathbf{x}'$ and define the operation, with the unit quaternion q, as

$$x = q \circ x' \circ q^{-1} = q \circ x' \circ \overline{q}, \qquad (25)$$

then this transformation, from x' to x, represents a rotation. The resulting quaternion x is a vectorial quaternion with the same length as x'. The case of reflection, the other possibility, can be excluded. The rotation matrix $\mathbf{R}$ in terms of the unit quaternions q can be derived from equation (25) as

$$\mathbf{x} = (q_0^2 - \mathbf{q}\cdot\mathbf{q})\mathbf{x}' + 2q_0(\mathbf{q}\times\mathbf{x}') + 2(\mathbf{q}\cdot\mathbf{x}')\mathbf{q} = \mathbf{R}\mathbf{x}' \qquad (26)$$

with

$$\mathbf{R} = \begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_2q_1+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_3q_1-q_0q_2) & 2(q_3q_2+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix}. \qquad (27)$$

This rotation matrix can also be written with the help of the quaternion matrix representation according to

$$\begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R} \end{pmatrix} = \mathbf{Q}\overline{\mathbf{Q}}^T = \overline{\mathbf{Q}}^T\mathbf{Q}. \qquad (28)$$

The quaternion q in the rotation matrix $\mathbf{R}$ according to equation (27), is identified as the set of Euler parameters for the description of finite rotation. According to Euler's theorem on finite rotation, a rotation in space can always be described by a rotation along a certain axis over a certain angle. With the unit vector $\mathbf{e}_\mu$ representing the axis and the angle of rotation μ, right-handed positive, the Euler parameters q can be interpreted as

$$q_0 = \cos(\mu/2) \quad \text{and} \quad \mathbf{q} = \sin(\mu/2)\mathbf{e}_\mu. \qquad (29)$$

Since the Euler parameters are unit quaternions the subsidiary condition,

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1, \qquad (30)$$

must always be satisfied. The quaternion x' in (25) can now be associated with the algebraic components of a vector in a body fixed frame and the quaternion x as the corresponding components expressed in a space fixed frame.

The Euler parameters q for successive rotations, where the sequence of rotations are described by the Euler parameters r and p, are given by their quaternion product $q = p \circ r$. This property can successfully be used if one knows the initial rotation p and the final configuration q and needs to calculate the relative rotation r. Simple quaternion calculus gives us $r = \overline{p} \circ q$.

Before we derive the rotational equations of motion for a spatial rigid body in terms of Euler parameters we have to express the angular velocities and accelerations in terms of the Euler parameters and its time derivatives. By differentiation of the rotational transformation (25) as in

$$\dot{x} = \dot{q} \circ x' \circ \overline{q} + q \circ x' \circ \dot{\overline{q}}, \qquad (31)$$

and substitution of the body fixed coordinates according to $x' = \overline{q} \circ x \circ q$, realizing that $\overline{q} \circ q$ is the unit element $(1, \mathbf{0})$, the velocity reads

$$\dot{x} = \dot{q} \circ \overline{q} \circ x + x \circ q \circ \dot{\overline{q}}. \qquad (32)$$

The scalar part of the products $\dot{q} \circ \overline{q}$ and $q \circ \dot{\overline{q}}$ are zero, since q is a unit quaternion, and the vector parts are opposite so we may write: $\dot{q} \circ \overline{q} = (0, \mathbf{w})$ and $q \circ \dot{\overline{q}} = (0, -\mathbf{w})$. The velocity $\dot{x}$ now has a zero scalar part, as expected, and a vectorial part, $\dot{\mathbf{x}} = 2\mathbf{w} \times \mathbf{x}$, so $\boldsymbol{\omega} = 2\mathbf{w}$. We conclude that the angular velocity $\boldsymbol{\omega}$ expressed in the space fixed reference in terms of the Euler parameters q and its time derivatives is given by

$$\omega = 2\dot{q} \circ \overline{q} \quad \text{or} \quad \begin{pmatrix} 0 \\ \boldsymbol{\omega} \end{pmatrix} = 2\overline{\mathbf{Q}}^T \begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix}. \qquad (33)$$

The inverse, the time derivatives $\dot{q}$ of the Euler parameters for given q and ω, can be found as

$$\dot{q} = \frac{1}{2}\omega \circ q \quad \text{or} \quad \begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix} = \frac{1}{2}\overline{\mathbf{Q}} \begin{pmatrix} 0 \\ \boldsymbol{\omega} \end{pmatrix}. \qquad (34)$$

Note that these time derivatives are always uniquely defined, opposed to Euler angles or any other classical combination of 3 parameters for describing spatial rotation like for example Rodrigues parameters or Cardan angles. The angular velocities $\boldsymbol{\omega}'$ expressed in a body fixed reference frame can be derived in the same manner, or by application of the rotational transformation (28), as

$$\omega' = 2\overline{q} \circ \dot{q} \quad \text{or} \quad \begin{pmatrix} 0 \\ \boldsymbol{\omega}' \end{pmatrix} = 2\mathbf{Q}^T \begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix}, \tag{35}$$

and with the inverse

$$\dot{q} = \frac{1}{2} q \circ \omega' \quad \text{or} \quad \begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix} = \frac{1}{2}\mathbf{Q} \begin{pmatrix} 0 \\ \boldsymbol{\omega}' \end{pmatrix}. \tag{36}$$

The angular accelarations are found by differentiation of the expressions for ω and ω', resulting in

$$\begin{pmatrix} 0 \\ \dot{\boldsymbol{\omega}} \end{pmatrix} = 2\overline{\mathbf{Q}}^T \begin{pmatrix} \ddot{q}_0 \\ \ddot{\mathbf{q}} \end{pmatrix} + 2 \begin{pmatrix} |\dot{q}|^2 \\ \mathbf{0} \end{pmatrix}, \tag{37}$$

and expressed in the body fixed reference frame

$$\begin{pmatrix} 0 \\ \dot{\boldsymbol{\omega}}' \end{pmatrix} = 2\mathbf{Q}^T \begin{pmatrix} \ddot{q}_0 \\ \ddot{\mathbf{q}} \end{pmatrix} + 2 \begin{pmatrix} |\dot{q}|^2 \\ \mathbf{0} \end{pmatrix}. \tag{38}$$

The inverse, the second order time derivatives $\ddot{q}$ of the Euler parameters in terms of q, $\dot{q}$ and $\dot{\omega}$, goes without saying.

The equations of motion for the rotation of a rigid body in a space (17) can be expressed in terms of Euler parameters and its time derivatives by application of the principle of virtual power and introduction of the Lagrangian multiplier λ for the norm constraint (30) written as

$$\Phi = q_0^2 + q_1^2 + q_2^2 + q_3^2 - 1 = 0, \tag{39}$$

resulting in the virtual power equation for a rigid body as

$$(\mathbf{M}' - \mathbf{J}'\dot{\boldsymbol{\omega}}' - \boldsymbol{\omega}' \times (\mathbf{J}'\boldsymbol{\omega}'))^T \delta\boldsymbol{\omega}' = \lambda \delta\dot{\Phi}. \tag{40}$$

The virtual constraint rate can be derived from (39) as

$$\delta\dot{\Phi} = 2q_0\delta\dot{q}_0 + 2\mathbf{q}^T\delta\dot{\mathbf{q}}. \tag{41}$$

The equations of motion can be obtained by substitution of the virtual constraint rates (41) and the angular velocities (35) and accelerations (38) in the virtual power equation (40). Assuming arbitrary virtual Euler parameter velocities $(\delta\dot{q}_0, \delta\dot{\mathbf{q}})$ and adding the constraints on the accelerations of the Euler parameters from (37) or (38) yields

$$\begin{bmatrix} 4\mathbf{Q}\begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{J}' \end{pmatrix}\mathbf{Q}^T & 2\begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} \\ 2(q_0, \mathbf{q}^T) & 0 \end{bmatrix} \begin{bmatrix} \ddot{q}_0 \\ \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} 2\mathbf{Q}\begin{pmatrix} 0 \\ \mathbf{M}' \end{pmatrix} + 8\dot{\mathbf{Q}}\begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{J}' \end{pmatrix}\dot{\mathbf{Q}}^T\begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} \\ -2|\dot{q}|^2 \end{bmatrix}. \tag{42}$$

These are the constrained equations of motion for a single rigid body expressed in terms of Euler parameters. The multiplier λ can for this single body be obtained by premultiplying the first four equations by $(q_0, \mathbf{q})^T$ and is indentified as twice the rotational kinetic energy of the body

$$\lambda = 4\begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix}^T \dot{\mathbf{Q}}\begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{J}' \end{pmatrix}\dot{\mathbf{Q}}^T\begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix} = \boldsymbol{\omega}'^T\mathbf{J}'\boldsymbol{\omega}'. \tag{43}$$

The transformations of an applied torque, body fixed $\mathbf{M}'$ or space fixed $\mathbf{M}$, to the torque parameters $(f_0, \mathbf{f})$, which are dual to the Euler parameters, are apparently

$$\begin{pmatrix} f_0 \\ \mathbf{f} \end{pmatrix} = 2\mathbf{Q}\begin{pmatrix} 0 \\ \mathbf{M}' \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} f_0 \\ \mathbf{f} \end{pmatrix} = 2\overline{\mathbf{Q}}\begin{pmatrix} 0 \\ \mathbf{M} \end{pmatrix}. \tag{44}$$

Again, as in the case of the Euler angles, the equations of motion need not always be transformed into Euler parameters and their time derivatives. It is computationally far more efficient to calculate the motion of a rigid body using the angular velocities $\boldsymbol{\omega}'$ together with the Euler parameters $q = (q_0, \mathbf{q})$ as the state variables. The state equations then become

$$\dot{\boldsymbol{\omega}}' = \mathbf{J}'^{-1}[\mathbf{M}' - \boldsymbol{\omega}' \times (\mathbf{J}'\boldsymbol{\omega}')], \tag{45}$$

$$\begin{pmatrix} \dot{q}_0 \\ \dot{\mathbf{q}} \end{pmatrix} = \frac{1}{2}\mathbf{Q}\begin{pmatrix} 0 \\ \boldsymbol{\omega}' \end{pmatrix}. \tag{46}$$

Numerical integration of these equations will lead to errors in the constraint equation (39) which can be resolved by renormalising the Euler parameters, as in $q = q/|q|$. This is known as the coordinate projection method and if preformed after each numerical integration step proves to be accurate and stable [11].

The use of Euler parameters within the general purpose multibody dynamics software package SPACAR [12] has, over the years, proved to be a success mainly due to the singularity-free and fast calculation of rotational motion.

3 Conclusions

Drawing rotational motion by a pair of cans in series leads to unambiguous interpretation of the rational motion. Euler parameters lead to singularity-free and fast calculation of rotational motion. Application of quaternion algebra eases the derivation of the necessary expressions.

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