tangent is not zero and the height has no stationary value. That condition is no longer required because the ball arrived at the limit of the available configuration space and there the variation of the position is not reversible.

> *Summary.* The extremum of a function requires a stationary value only for reversible displacements. On the boundary of the configuration space, where the variation of the position is not reversible, an extremum is possible without a stationary value.

5. Auxiliary conditions. The Lagrangian λ-method. The problem of minimizing a function does not always present itself in the form considered above. The configuration space in which the point P can move may be restricted to less than n dimensions by certain kinematical relations which exist between the coordinates. Such kinematical conditions are called "auxiliary conditions" of the given variation problem. If such conditions do not exist and the variables $u_1, \ldots, u_n$ can be varied without restriction, we have a "free" variation problem, as considered previously (sections 1-3).

We shall now investigate the variation of the function

$$F = F(u_1, u_2, \ldots, u_n), \tag{25.1}$$

with the auxiliary condition

$$f(u_1, u_2, \ldots, u_n) = 0. \tag{25.2}$$

Our first thought would be to eliminate one of the u_k—for example u_n—from the auxiliary condition, expressing it in terms of the other u_k. Then our function would depend on the n-1 unrestricted variables $u_1, \ldots, u_{n-1}$ and could be handled as a free variation problem. This method is entirely justified and sometimes advisable. But frequently the elimination is a rather cumbersome procedure. Moreover, the condition (25.2) may be symmetric in the variables $u_1, \ldots, u_n$ and there would be no reason why one of the variables should be artificially designated as dependent, the others as independent variables.

Lagrange devised a beautiful method for handling auxiliary conditions, the "method of the undetermined multiplier," which preserves the symmetry of the variables without eliminations, and still reduces the problem to one of free variation. The method works quite generally for any number of auxiliary conditions and is applicable even to non-holonomic conditions which are given as non-integrable relations between the differentials of the variables, and not as relations between the variables themselves.

In order to understand the nature of the Lagrangian multiplier method, we start with a single auxiliary condition, given in the form (25.2). Taking the variation of this equation we obtain the following relation between the δu_k:

$$\delta f = \frac{\partial f}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial f}{\partial u_n}\,\delta u_n = 0\ ; \qquad (25.3)$$

while the fact that the variation of F has to vanish at a stationary value, gives

$$\delta F = \frac{\partial F}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial F}{\partial u_n}\,\delta u_n = 0. \qquad (25.4)$$

We know from section 3 that the condition (25.4) would lead to the vanishing of each $\frac{\partial F}{\partial u_k}$ if the δu_k were all independent of each other. This, however, is not the case, because of the condition (25.3). We agree to eliminate δu_n in terms of the other variations—assuming that $\frac{\partial f}{\partial u_n}$ is not zero at the point P—and then consider the other δu_k as free variations. But before we do so, we shall modify the expression (25.4). It is obviously permissible to multiply the left-hand side of (25.3) by some undetermined factor λ, which is a function of $u_1, \ldots, u_n$, and add it to δF. This does not change the value of δF at all since we have added zero. Hence it is still true that:

$$\frac{\partial F}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial F}{\partial u_n}\,u\delta_n + \lambda\left(\frac{\partial f}{\partial u_1}\delta u_1 + \ldots + \frac{\partial f}{\partial u_n}\delta u_n\right) = 0. \qquad (25.5)$$

This move is not trivial because, although we have added zero, we have actually added a *sum*; the individual terms of the sum are not zero, only the total sum is zero.

We write (25.5) in the form

$$\sum_{k=1}^{n} \left(\frac{\partial F}{\partial u_k} + \lambda \frac{\partial f}{\partial u_k}\right) \delta u_k = 0. \tag{25.6}$$

We wish to eliminate δu_n. But now we can choose λ so that *the factor multiplying* δu_n *shall vanish*:

$$\frac{\partial F}{\partial u_n} + \lambda \frac{\partial f}{\partial u_n} = 0. \tag{25.7}$$

This dispenses with the task of eliminating δu_n. After that our sum is reduced to only n-1 terms:

$$\sum_{k=1}^{n-1} \left(\frac{\partial F}{\partial u_k} + \lambda \frac{\partial f}{\partial u_k}\right) \delta u_k = 0, \tag{25.8}$$

and since only those δu_k remain which can be chosen arbitrarily, the conditions of a free variation problem are applicable. These require that the coefficient of each δu_k shall vanish:

$$\frac{\partial F}{\partial u_k} + \lambda \frac{\partial f}{\partial u_k} = 0, \quad (k = 1, 2, \ldots, n-1). \tag{25.9}$$

The conditions (25.9), combined with the condition (25.7) on λ, lead to the conclusion that *each coefficient of the sum* (25.6) *vanishes, just as if all the variations* δu_k *were free variations.* The result of Lagrange's "method of the undetermined multiplier" can be formulated thus: instead of considering the vanishing of δF, consider the vanishing of

$$\delta F + \lambda \delta f, \tag{25.10}$$

and drop the auxiliary condition, handling the u_k as free, independent variables.

We have

$$\delta F + \lambda \delta f = \delta(F + \lambda f), \tag{25.11}$$

since the term $f\delta\lambda$ vanishes on account of the auxiliary condition $f = 0$. Hence we can express the result of our deductions in an even more striking form. Instead of putting the first variation of F equal to zero, modify the function F to

$$\overline{F} = F + \lambda f, \tag{25.12}$$

and put its first variation equal to zero, for *arbitrary* variations of the u_k.

We generalize this λ-method for the case of an arbitrary number of auxiliary conditions. Let us assume once more that the stationary value of F is sought, but under m independent restricting conditions:

$$\begin{aligned} f_1\ (u_1, u_2, \ldots, u_n) &= 0, \\ &\cdots \\ &\cdots \qquad (m < n). \\ &\cdots \\ f_m(u_1, u_2, \ldots, u_n) &= 0. \end{aligned} \tag{25.13}$$

These auxiliary conditions establish the following relations between the variations δu_k:

$$\begin{aligned} \delta f_1 &= \frac{\partial f_1}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial f_1}{\partial u_n}\,\delta u_n = 0, \\ &\cdots \\ &\cdots \\ &\cdots \\ \delta f_m &= \frac{\partial f_m}{\partial u_1}\,\delta u_1 + \ldots + \frac{\partial f_m}{\partial u_n}\,\delta u_n = 0. \end{aligned} \tag{25.14}$$

Because of these conditions m of the δu_k can be designated as dependent variables and expressed in terms of the others. We shall consider the *last* m of the u_k as dependent, the first $n - m$ as independent, variables.

Now the given variational problem requires the vanishing of

$$\delta F = \sum_{k=1}^{n} \frac{\partial F}{\partial u_k}\,\delta u_k \tag{25.15}$$

for all possible variations δu_k which satisfy the given auxiliary conditions. We should express the last m δu_k in terms of the independent δu_k. However, before doing so let us modify the expression (25.15) by adding the left-hand sides of the equations (25.14) after multiplying each one by some undetermined λ-factor. We thus get

$$\sum_{k=1}^{n} \left(\frac{\partial F}{\partial u_k} + \lambda_1 \frac{\partial f_1}{\partial u_k} + \ldots + \lambda_m \frac{\partial f_m}{\partial u_k}\right) \delta u_k = 0. \tag{25.16}$$

Now the elimination of the last m δu_k can be accomplished by the proper choice of the λ-factors, so that

$$\frac{\partial F}{\partial u_k} + \lambda_1 \frac{\partial f_1}{\partial u_k} + \ldots + \lambda_m \frac{\partial f_m}{\partial u_k} = 0, \quad (k = n - m + 1, \ldots, n). \quad (25.17)$$

This leaves

$$\sum_{k=1}^{n-m} \left(\frac{\partial F}{\partial u_k} + \lambda_1 \frac{\partial f_1}{\partial u_k} + \ldots + \lambda_m \frac{\partial f_m}{\partial u_k} \right) \delta u_k = 0. \quad (25.18)$$

But all the δu_k which remain in (25.18) are *free* variations. Hence the coefficient of each δu_k must vanish separately. In the final analysis we have the equations

$$\frac{\partial F}{\partial u_k} + \lambda_1 \frac{\partial f_1}{\partial u_k} + \ldots + \lambda_m \frac{\partial f_m}{\partial u_k} = 0, \ (k = 1, \ldots, n), \quad (25.19)$$

which can be considered as obtained from the variational principle

$$\delta F + \lambda_1 \, \delta f_1 + \ldots + \lambda_m \, \delta f_m = 0, \quad (25.20)$$

considering *all* the u_k as independent variables. Thus in the final result the distinction between dependent and independent variables disappears.

Equation (25.20) can be stated even more strikingly by writing it in the form

$$\delta(F + \lambda_1 f_1 + \ldots + \lambda_m f_m) = 0, \quad (25.21)$$

and interpreting this equation as follows: *Instead of asking for the stationary value of F, we ask for the stationary value of the modified function*

$$\bar{F} = F + \lambda f_1 + \ldots + \lambda_m f_m, \quad (25.22)$$

dropping the auxiliary conditions and handling this as a free variation problem. This yields n equations. In addition to these equations we have to satisfy the m auxiliary conditions (25.13). This gives $n + m$ equations for the $n + m$ unknowns

$$u_1, u_2, \ldots, u_n; \ \lambda_1, \lambda_2, \ldots, \lambda_m. \quad (25.23)$$

The ingenious multiplier-method of Lagrange changes a problem of $n - m$ degrees of freedom to a problem of $n + m$

degrees of freedom. If we add to the n variables u_k the m quantities λ_i as additional variables and ask for the stationary value of the function $\bar{F}$, this variation problem gives the same n equations as we had before if we vary with respect to the u_k, while the variations of the λ_i give the m additional conditions

$$f_1 = 0, \ldots, f_m = 0. \tag{25.24}$$

These are exactly the given auxiliary conditions, but now obtained *a posteriori*, on account of the variation problem.

The method of Lagrange permits the use of surplus coordinates—a great convenience in many considerations of mechanics. It preserves the full symmetry of all coordinates by making it unnecessary to distinguish between dependent and independent variables.

Summary. The Lagrangian-multiplier method reduces a variation problem with auxiliary conditions to a free variation problem without auxiliary conditions. We modify the given function F, which is to be made stationary, by adding the left-hand sides of the auxiliary conditions, after multiplying each by an undetermined factor λ. Then we handle the modified problem as a free variation problem. The resulting conditions, together with the given auxiliary conditions, determine the unknowns and the λ-factors.

6. Non-holonomic auxiliary conditions. As was pointed out in chap. I, section 6, the restrictions on the mechanical variables of a problem may be given in a differential instead of a finite form. We then have a variation problem with non-holonomic auxiliary conditions. The equations (25.13) do not exist in this case, but we have relations analogous to the *differentiated* forms (25.14) of the auxiliary conditions. The only difference is that the left-hand sides of these equations are no longer exact differentials but merely infinitesimal quantities. We can write the non-holonomic conditions in the following form:

$$\begin{aligned} \overline{\delta f}_1 &= A_{11}\delta u_1 + A_{12}\delta u_2 + \ldots + A_{1n}\delta u_n = 0, \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ \overline{\delta f}_m &= A_{m1}\delta u_1 + A_{m2}\delta u_2 + \ldots + A_{mn}\delta u_n = 0. \end{aligned} \tag{26.1}$$

Here the A_{ik} are given functions of the u_i which cannot be considered as the partial derivatives of a function f_i.

Non-holonomic conditions cannot be handled by the elimination method, because the equations for eliminating some variables as dependent variables do not exist. The Lagrangian λ-method, however, is again available. By exactly the same procedure as before, we can obtain an equation analogous to (25.20), namely:

$$\delta F + \lambda_1 \overline{\delta f}_1 + \ldots + \lambda_m \overline{\delta f}_m = 0 \; ; \tag{26.2}$$

and again all the δu_k are handled as free variations. The only difference lies in the fact that we cannot proceed to the equation (25.21) and have to be content with the differential formulation of the procedure. The reduction of a conditioned variation problem to a free variation problem is once more accomplished.

Summary. The Lagrangian λ-method is applicable even to non-holonomic conditions. We multiply the left sides of these conditions by some undetermined λ-factors and add them to the variation of the function F which is to be made stationary. This whole expression is put equal to zero, considering all the variations δu_k as free variations.

7. The stationary value of a definite integral. The analytical problems of motion involve a special type of extremum problem: the stationary value of a *definite integral.* The branch of mathematics dealing with problems of this nature is called the Calculus of Variations. A typical problem of this kind is that of the brachistochrone (the curve of quickest descent), first formulated and solved by John Bernoulli (1696); it is one of the earliest instances of a variational problem. We wish to find a

CHAPTER III

THE PRINCIPLE OF VIRTUAL WORK

1. The principle of virtual work for reversible displacements. The first variational principle we encounter in the science of mechanics is the principle of virtual work. It controls the equilibrium of a mechanical system and is fundamental for the later development of analytical mechanics.

In the Newtonian form of mechanics a particle is in equilibrium if the resulting force acting on that particle is zero. This form of mechanics isolates the particle and replaces all constraints by forces. The inconvenience of this procedure is obvious if we think of such a simple problem as the equilibrium of a lever. The lever is composed of an infinity of particles and an infinity of inner forces acting between them. The analytical treatment can dispense with all these forces and take only the external force—i.e. in this case the force of gravity—into account. This is accomplished by performing only such virtual displacements as are in harmony with the given constraints. In the case of the lever, for example, we let the lever rotate around its fulcrum *as a rigid body*, thus preserving the mutual distance of any two particles. By this procedure the inner forces which produce the constraints need not be considered.

The mechanical behavior of a rigid body is certainly very different from that of a sandpile. There are strong inner forces acting between the particles of a rigid body which keep these particles together and which do not act between the particles of a sandpile. But how can we prove the presence of these forces? By trying to *break* the rigid body, i.e. by trying to move the particles relative to each other in a manner which is *not* in harmony with the given constraints. If we merely move a rigid body or a sandpile by rotation and translation, the mechanical difference between the two systems disappears, because now the strong inner forces which characterize the rigid body in contrast to the sandpile do not come into action. This is the reason why in the variational treatment of mechanics the "forces of constraint" which maintain certain given kinematical conditions are neglected, and only the work of the "impressed forces" needs to be taken into account. We may eliminate the action of the inner forces, since the virtual displacements applied to the system are in harmony with the given kinematical conditions. The number of equations obtained by this procedure is smaller than the number of particles. But it is exactly equal to the number of degrees of freedom which characterize the system.

Let us use at first the language of vectorial mechanics. We assume that the given external forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act at the points $P_1, P_2, \ldots, P_n$ of the system. The virtual displacements of these points will be denoted by

$$\delta\mathbf{R}_1,\ \delta\mathbf{R}_2, \ldots, \delta\mathbf{R}_n. \tag{31.1}$$

These virtual displacements must be in harmony with the given kinematical constraints, and we shall assume that they are *reversible*, i.e. the given constraints do not prevent us from changing an arbitrary $\delta\mathbf{R}_i$ into $-\delta\mathbf{R}_i$.

Now the principle of virtual work asserts that *the given mechanical system will be in equilibrium if, and only if, the total virtual work of all the impressed forces vanishes*:

$$\overline{\delta w} = \mathbf{F}_1 . \delta\mathbf{R}_1 + \mathbf{F}_2 . \delta\mathbf{R}_2 + \ldots + \mathbf{F}_n . \delta\mathbf{R}_n = 0. \tag{31.2}$$

Let us translate this equation into analytical language. For this purpose we express the rectangular coordinates x_i, y_i, z_i as functions of the generalized coordinates $q_1, q_2, \ldots, q_n$, exactly as we have done in chap. I, section 7. The differential form (31.2) is then transformed into the new differential form

$$\overline{\delta w} = F_1\delta q_1 + F_2\delta q_2 + \ldots + F_n\delta q_n, \tag{31.3}$$

where $F_1, F_2, \ldots, F_n$ are called the components of the generalized force. They form a vector of the n-dimensional configuration space.

The principle of virtual work requires that

$$F_1\delta q_1 + F_2\delta q_2 + \ldots + F_n\delta q_n = 0. \tag{31.4}$$

We can give a striking geometrical interpretation of this equation. The left-hand side of the equation is nothing but the "scalar product" of force and virtual displacement. The vanishing of this scalar product means that *the force F_i is perpendicular to any possible virtual displacement.*

Let us assume at first that the given mechanical system is free of any constraints. In that case the *C*-point of the configuration space can be displaced in an arbitrary direction. Then the principle (31.4) requires that the force F_i shall vanish, because there is no vector which can be perpendicular to all directions in space.

Let us assume that the *C*-point has to stay within a certain $(n - m)$-dimensional *subspace* of the configuration space, on account of m given kinematical constraints. Then the condition (31.2) no longer requires the vanishing of the force F_i, but only its *perpendicularity* to that subspace. This amounts to $n - m$ equations, in conformity with the $n - m$ degrees of freedom of the mechanical system.

We now come to the *physical* interpretation of the principle of virtual work. According to Newtonian mechanics, the state of equilibrium requires that the resultant force acting on *any* particle of the system shall vanish. This resultant force is the sum of the impressed force and the forces which maintain the given constraints. These latter forces are usually called "forces of reaction." Since the principle of equilibrium requires that "impressed force plus resultant force of reaction equals zero," we see that the virtual work of the impressed forces can be replaced by the negative virtual work of the forces of reaction. Hence the principle of virtual work can be formulated in the following form, which we shall call Postulate *A*:

> "The virtual work of the forces of reaction is always zero for any virtual displacement which is in harmony with the given kinematic constraints."

This postulate is not restricted to the realm of statics. It applies equally to dynamics, when the principle of virtual work is suitably generalized by means of d'Alembert's principle. Since all the fundamental variational principles of mechanics, the principles of Euler, Lagrange, Jacobi, Hamilton, are but alternative mathematical formulations of d'Alembert's principle, Postulate A is actually the *only* postulate of analytical mechanics, and is thus of fundamental importance.[1]

The principle of least action assumes a special significance in the particularly important case where the impressed force F_i is monogenic, i.e. derivable from a single scalar function, the work function. In this case the virtual work is equal to the variation of the work function $U(q_1, \ldots, q_n)$. Since the work function can be replaced by the negative of the potential energy, we can say that the state of equilibrium of a mechanical system is distinguished by the stationary value of the potential energy, i.e. by the condition

$$\delta V = 0. \tag{31.5}$$

If the equilibrium is *stable*, the potential energy must assume its minimum value—the minimum understood in the local sense—while in general, equilibrium does not require the minimum, but only the stationary value, of V.

> *Summary.* The principle of virtual work demands that for the state of equilibrium the work of the impressed forces is zero for any infinitesimal variation of the configuration of the system which is in harmony with the given kinematical constraints. For monogenic forces, this leads to the condition that, for equilibrium, the potential energy shall be stationary with respect to all kinematically permissible variations.

The following two sections will be devoted to the application of the principle of virtual work to the statics of a rigid body. The results are well known

[1]Those scientists who claim that analytical mechanics is nothing but a mathematically different formulation of the laws of Newton must assume that Postulate A is deducible from the Newtonian laws of motion. The author is unable to see how this can be done. Certainly the third law of motion, "action equals reaction," is not wide enough to replace Postulate A.

from elementary vectorial mechanics, but their deduction from one fundamental principle is a valuable experience.

2. The equilibrium of a rigid body. A rigid body which can move freely in space has six degrees of freedom: three on account of translation and three on account of rotation. Making use of the superposition principle of infinitesimal quantities, we can apply these two types of displacements independently of one another.

(A) *Translation.* An infinitesimal translation produces at each point of the rigid body the same displacement. Let ϵ be the extent of the infinitesimal displacement, and $\mathbf{B}$ a vector of unit length. We then have for the virtual displacement $\delta\mathbf{R}_k$ of the particle P_k

$$\delta\mathbf{R}_k = \epsilon\mathbf{B}, \tag{32.1}$$

and the resultant work becomes

$$\overline{\delta w} = \Sigma(\mathbf{F}_k \cdot \epsilon\mathbf{B}) = \epsilon\mathbf{B} \cdot \Sigma\mathbf{F}_k. \tag{32.2}$$

Since the vector B can be chosen in any direction, the vanishing of (32.2) requires:

$$\overline{\mathbf{F}} = \Sigma\mathbf{F}_k = 0, \tag{32.3}$$

which means that the *resultant force* $\overline{\mathbf{F}}$ *of all the impressed forces vanishes.*

A piston may be forced to move up and down in its tube. The vector $\mathbf{B}$ has then a definite direction, and the vanishing of (32.3) requires only that the *component of* $\overline{\mathbf{F}}$ *in the direction of motion shall vanish.*

(B) *Rotation.* Let ϵ be the angle of an infinitesimal rotation, and Ω a vector of unit length along the axis of rotation. The displacement of the point P_k due to the rotation can be written as follows:

$$\delta\mathbf{R}_k = \epsilon\Omega \times \mathbf{R}_k, \tag{32.4}$$

where $\mathbf{R}_k$ denotes the position vector of P_k with respect to an origin on the axis of rotation.

The work of the force $\mathbf{F}_k$ becomes

$$\overline{\delta w}_k = \mathbf{F}_k \cdot \epsilon\Omega \times \mathbf{R}_k = \epsilon\Omega \cdot (\mathbf{R}_k \times \mathbf{F}_k) = \epsilon\Omega \cdot \mathbf{M}_k, \tag{32.5}$$

where we have introduced the vector

$$\mathbf{M}_k = \mathbf{R}_k \times \mathbf{F}_k \tag{32.6}$$

to denote the "moment of the force" about the origin.