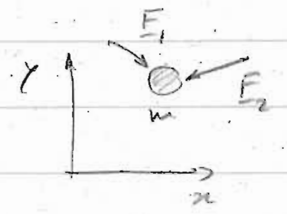
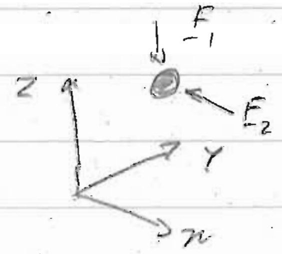
From 2D two 3D

We want to know the Newton-Euler Equations in 3D for a Rigid Body?
The Newton part is easy, just a point mass



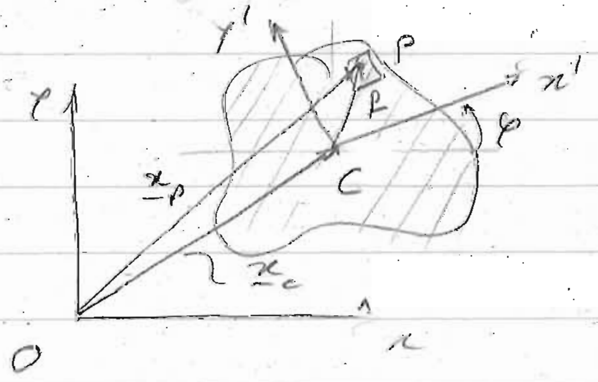
2D → 3D



$$\sum F_{ix} = m\ddot{x}$$
$$\sum F_{iy} = m\ddot{y} \quad \rightarrow \sum \underline{F}_i = m\underline{\ddot{x}}$$

$$\sum F_{ix} = m\ddot{x}$$
$$\sum F_{iy} = m\ddot{y} \quad \Rightarrow \sum \underline{F}_i = m\underline{\ddot{x}}$$
$$\sum F_{iz} = m\ddot{z}$$

The Euler part is about Rotation. Recap 2D:



Position cm (x_c, y_c)
Orientation φ

C-x'y' coordinate system is fixed to the rigid body → algebraic components of $\underline{r}$ are constant in this coordinate system.
Finding p in the frame of reference, the O-xy coordinate system

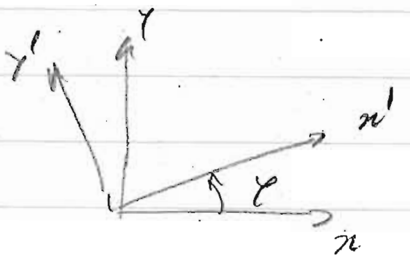
$$\underline{x}_p = \underline{x}_c + \underline{x}_{p/c}$$

But $\underline{x}'_{prc}$ is constant, rigid body

$$\underline{x}_p = \underline{x}_c + R(\varphi)\,\underline{x}'_{prc}$$

Any point on the rigid body can be found from a translation $\underline{x}_c$ and a rotation φ.

The Rotation matrix in 2D is $R(\varphi) = \begin{pmatrix} c_\varphi & -s_\varphi \\ s_\varphi & c_\varphi \end{pmatrix}$



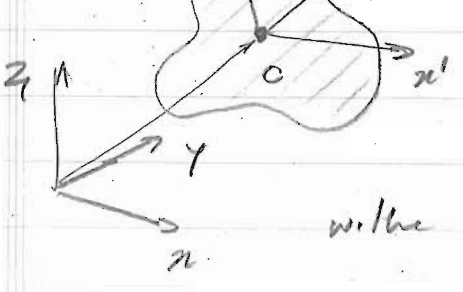
$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} c_\varphi & -s_\varphi \\ s_\varphi & c_\varphi \end{pmatrix}}_{R} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

After adding up all pointmasses in the Newton eqn. for a 2D point mass

$$\left.\begin{aligned} \Sigma F_x &= m\ddot{x}_c \\ \Sigma F_y &= m\ddot{y}_c \end{aligned}\right\} \text{Newton}$$
$$\Sigma M_c = J_c\ddot{\varphi} \quad \} \text{ Euler} \longrightarrow 2^*$$

with $J_c = \int_V (x'^2 + y'^2)\,dm$ Massmoment of Inertia

In 3D we get:

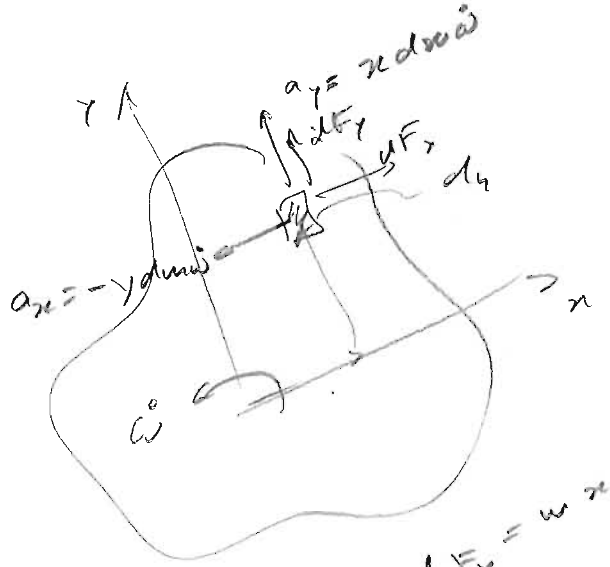


$$\Sigma \underline{F} = m\,\ddot{\underline{x}}_c \quad \text{Newton}$$

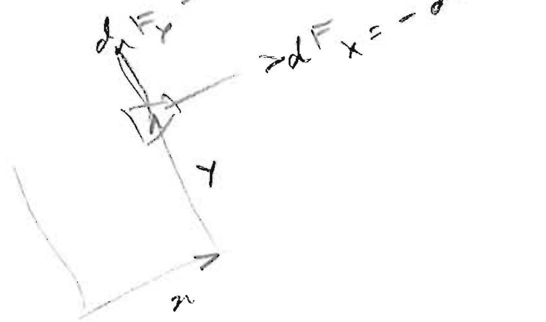
$$\Sigma \underline{M}_c = J_c\,\dot{\underline{\omega}} + \underline{\omega} \times (J_c\,\underline{\omega}) \quad \text{Euler}$$

with the angular velocity vector $\underline{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$

2*



$f = ma$



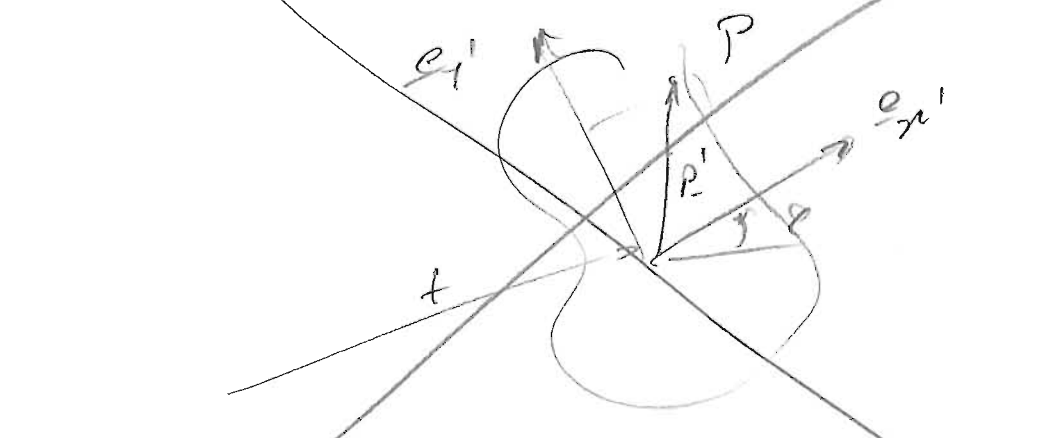
$$dM = x\,dF_y - y\,dF_x$$

$$= x\,x\,\dot\omega\,dm + y(-y\,\dot\omega\,dm)$$

$$dM = x^2 + y^2\,dm\,\dot\omega$$

$$M = \int (x^2+y^2)\,dm\,\dot\omega$$

$$M = I.\ddot\varphi$$



$$\underline{P} = \underline{t} + R.p'$$

and the mass moment of inertia matrix

$$J_c = \begin{pmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{pmatrix} \quad \text{with} \quad J_{xx} = \int (y^2+z^2)\,dm \quad ?? \qquad J_{xy} = -\int xy\,dm \quad ???$$

Or along the principal axis, you get rid of the off-diagonal terms:

$$J_c = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{pmatrix}$$

(J_x, J_y and J_z are the eigenvalues of J_c)

Back to Euler $\Sigma \underline{M}_c = J_c \dot{\underline{\omega}} + \underline{\omega} \times (J_c \underline{\omega})$
If the body rotates then J_c is not constant!
Better to express Euler in body fixed coordinate system C-$x'y'z'$

$$\Sigma \underline{M}'_c = J'_c \dot{\underline{\omega}}' + \underline{\omega}' \times (J'_c \underline{\omega}')$$

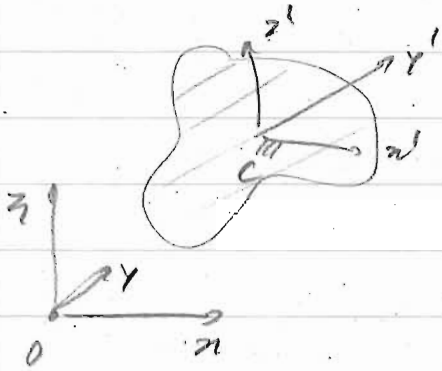
$\underline{\omega}'$ the angular velocity in the $Cx'y'z'$ coord system (local)

$\underline{M}'_c$ the moment in the local coord system

J'_c is now a constant matrix.

Summarize Newton-Euler in 3D

Newton-Euler eqn's of motion for a rigid body in 3D:



$$\Sigma F_x = m \ddot{x}_c$$
$$\Sigma F_y = m \ddot{y}_c$$
$$\Sigma F_z = m \ddot{z}_c$$

$$\Sigma M'_x = J'_x \dot{\omega}'_x - (J'_y - J'_z)\omega'_y \omega'_z$$
$$\Sigma M'_y = J'_y \dot{\omega}'_y - (J'_z - J'_x)\omega'_z \omega'_x$$
$$\Sigma M'_z = J'_z \dot{\omega}'_z - (J'_x - J'_y)\omega'_x \omega'_y$$

$Cx'y'z'$ body fixed coord system with principal axes

$$J' = \begin{pmatrix} J'_x & 0 & 0 \\ 0 & J'_y & 0 \\ 0 & 0 & J'_z \end{pmatrix}$$

These 3D Euler equations look strange and complex, but... 3D motion of rigid bodies can be strange and complex

For example: the stability of torque free 3D rotation of a rigid body (think of a satellite, a tennis racket or a remote control.)

Assume we start a rigid body with one big angular velocity and the other two zero $\underline{\omega} = (\omega_1, 0, 0)$. Will this stay this way or will the body start rotating along one of the other axis in other words is this rotation stable??

Everything in a body fixed coord system with principle axis along 1, 2, 3 axis. In centre of mass.
Assume and initial angular velocity $\underline{\omega}_0 = (\omega_1, 0, 0)$
The inertia matrix is $J_c = \mathrm{diag}(J_1, J_2, J_3)$
Torque free $\Sigma \underline{M} = \underline{0}$

Euler in local coord system: $\Sigma \underline{M} = J\dot{\underline{\omega}} + \underline{\omega} \times (J\underline{\omega})$

We assume small changes from the initial conf.

$$\Sigma \underline{M} = \underline{M}_0 + \Delta \underline{M}, \quad \underline{\omega} = \underline{\omega}_0 + \Delta\underline{\omega} \rightarrow \dot{\underline{\omega}} = \dot{\underline{\omega}}_0 + \Delta\dot{\underline{\omega}}$$

$$\underline{M}_0 + \Delta\underline{M} = J\cdot(\dot{\underline{\omega}}_0 + \Delta\dot{\underline{\omega}}) + (\underline{\omega}_0 + \Delta\underline{\omega}) \times (J\cdot(\underline{\omega}_0 + \Delta\underline{\omega}))$$

used these

$$\underline{M}_0 + \Delta\underline{M} = J\dot{\underline{\omega}}_0 + J\Delta\dot{\underline{\omega}} + \underline{\omega}_0 \times J\underline{\omega}_0 + \underline{\omega}_0 \times J\Delta\underline{\omega} + \Delta\underline{\omega} \times J\underline{\omega}_0 + \underbrace{\Delta\underline{\omega} \times J\Delta\underline{\omega}}_{O(\Delta^2)}$$

$\underline{\omega} = (\omega, 0, 0)$
$\Delta\underline{\omega} = (\alpha, \beta, \gamma)$
$J_c = (A, B, C)$

$$\underbrace{\underline{M}_0}_{0} = J\dot{\underline{\omega}}_0 + \underbrace{\underline{\omega}_0 \times J\underline{\omega}_0}_{0} \rightarrow \dot{\underline{\omega}}_0 = \underline{0} \ \& \ \underline{\omega}_0 \ \text{constant}$$

$$\Delta\underline{M} = J\Delta\dot{\underline{\omega}} + \underline{\omega}_0 \times J\Delta\underline{\omega} + \Delta\underline{\omega} \times J\underline{\omega}_0$$

$\Delta\underline{M} = \underline{0}$ We apply no extra Torques

$$\underline{0} = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{pmatrix}\begin{pmatrix} \Delta\dot{\omega}_1 \\ \Delta\dot{\omega}_2 \\ \Delta\dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} \omega_1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} J_1\Delta\omega_1 \\ J_2\Delta\omega_2 \\ J_3\Delta\omega_3 \end{pmatrix} + \begin{pmatrix} \Delta\omega_1 \\ \Delta\omega_2 \\ \Delta\omega_3 \end{pmatrix} \times \begin{pmatrix} J_1\omega_1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ -\omega_1 J_3 \Delta\omega_3 \\ \omega_1 J_2 \Delta\omega_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\omega_3 J_1 \omega_1 \\ -\Delta\omega_2 J_1 \omega_1 \end{pmatrix}$$

$$\underline{0} = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} \begin{pmatrix} \Delta\dot{\omega}_1 \\ \Delta\dot{\omega}_2 \\ \Delta\dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (J_1-J_3)\omega_1 \\ 0 & (J_2-J_1)\omega_1 & 0 \end{pmatrix} \begin{pmatrix} \Delta\omega_1 \\ \Delta\omega_2 \\ \Delta\omega_3 \end{pmatrix}$$

Assume a solution of the form $\Delta\underline{\omega} = \underline{a}\, e^{\lambda t}$

This leads to the eigenvalue problem

$$\begin{pmatrix} J_1\lambda & 0 & 0 \\ 0 & J_2\lambda & (J_1-J_3)\omega_1 \\ 0 & (J_2-J_1)\omega_1 & J_3\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot e^{\lambda t} = \underline{0} \quad \rightarrow$$

$$\mathrm{Det}=0 \rightarrow \quad J_1 J_2 J_3 \lambda^3 + 0 + 0 - 0 - (J_2-J_1)\omega_1 (J_1-J_3)\omega_1 \cdot J_1\lambda = 0$$

$$J_1\lambda \left\{ J_2 J_3 \lambda^2 - (J_1-J_3)(J_2-J_1)\omega_1^2 \right\} = 0$$

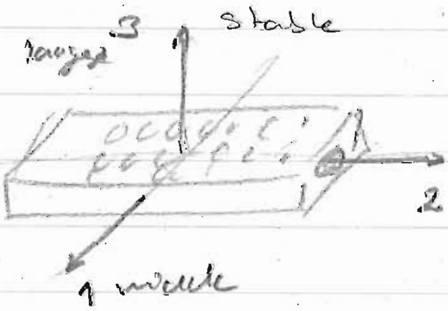
$$\rightarrow \lambda_1 = 0 \quad \text{and} \quad \lambda_{2,3} = \sqrt{\frac{(J_1-J_3)(J_2-J_1)}{J_2 J_3}}\ \omega_1$$

Unstable $\lambda_{2,3} > 0 \Longrightarrow J_1 > J_3$ and $J_2 > J_1 \Rightarrow J_2 > J_1 > J_3$

or $J_3 > J_1$ and $J_1 > J_2 \Rightarrow J_3 > J_1 > J_2$

Conclude rotation about the intermediate axis is unstable !!

Remote Control:



$J_3 > J_1 > J_2$

Tennis Racket