Lecture Notes
Multibody Dynamics B, wb1413

A. L. Schwab & Guido M.J. Delhaes

Laboratory for Engineering Mechanics
Mechanical Engineering
Delft University of Technolgy
The Netherlands

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Notes on the Bicycle Project

10.1 The Model

This is a recipe for making a full 3D dynamic model of a bicycle.

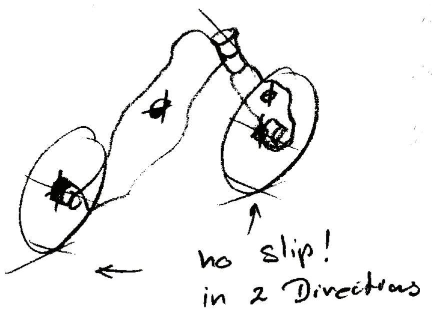

Figure 10.1: bicycle model

Start with some counting to see the number of degrees of freedom in the coordinate space and in the velocity space. The no-slip conditions are conditions in the velocities which can not be expressed in terms of the coordinates only. These type of constraints are called nonholonomic, opposed to the regular or holonomic constraints.

4	bodies	→	4×6	=	24	coordinates
3	hinges	→	3×5	=	15	constraints on coordinates, holonomic constraints
2	contact	→	2×1	=	2	constraints on coordinates, holonomic constraints
	points		2×2	=	4	constraints on velocities, nonholonomic constraints

So we have in total 24 coordinates, 17 constraints on the coordinates (and the velocities), and an additional 4 constraints on the velocities only. Which leaves $24 - 17 = 7$ degrees of freedom in the coordinates, and $24 - 21 = 3$ degrees of freedom in the velocities, the so-called mobility.

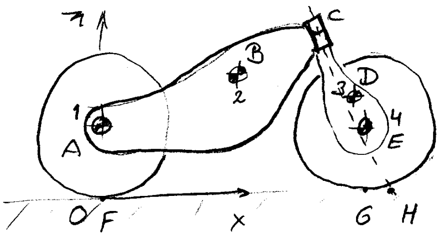

Figure 10.2: numbering of the bodies of the bicycle model

Look at the different bodies and number them. Pick a space fixed coordinate system $O-xyz$ located at the rear contact point. Label a number of interesting points $A, B, C, \cdots$.

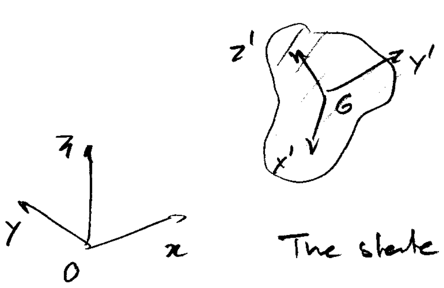

Figure 10.3: Unconstrained rigid body

The equations of motion for a single unconstrained rigid body are:

$$\begin{array}{l} \text{Newton } \rightarrow \Sigma \mathbf{f} = m\dot{\mathbf{v}}_G \\ \text{Euler } \rightarrow \Sigma \mathbf{M}'_G = \mathbf{J}'_G \dot{\boldsymbol{\omega}}' + \boldsymbol{\omega}' \times (\mathbf{J}'_G \boldsymbol{\omega}') \end{array}$$

where G is the centre of mass of the rigid body.

The state is described by the position and orientation of the centre of mass of the rigid body $(\mathbf{x}_G, \mathbf{p})$, where $\mathbf{p}$ are either Euler Angles or Cardan Angles or Euler parameters or $\cdots$, and the velocities of the centre of mass $(\mathbf{v}_G, \boldsymbol{\omega}')$, where the angular velocities $\boldsymbol{\omega}'$ are expressed in the body fixed coordinate system.

For the description of the orientation of the rigid bodies we will use the following $z - x - y$ or 3-1-2 Euler Angles denoted by $\mathbf{p} = (\phi, \theta, \psi)$, and where the recipe is given by,

ϕ rotate about $z-$ axis: heading or yaw
θ rotate about the rotated $x-$ axis: bank or lean
ψ rotate about the rotated $y-$ axis: pitch

but are of course best be depicted by the cans-in-series, see Figure 10.4

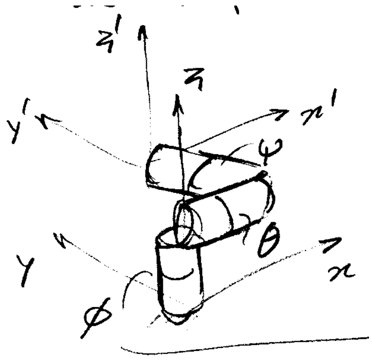

Figure 10.4: Definition Euler Angles by cans-in-series

Now why this choice? Look at a wheel, Figure 10.5. The yaw motion can be large so we put this rotation at the end or the beginning, in this case the beginning. The singular configuration is at a lean angle of $\theta = \pm 90^o$ which we try to avoid in cycling, so that will be the middle hinge.

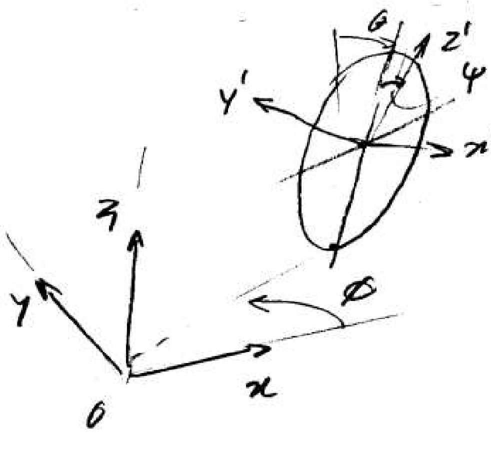

Figure 10.5: Definition of the coordinate system of a running wheel

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First derive the rotation matrices for this definition and the angular velocities.

$$\mathbf{x} = \mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi \mathbf{x}'$$

with: $\mathbf{R}_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{R}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta)0 \end{pmatrix}$ and $\mathbf{R}_\psi = \begin{pmatrix} \cos(\psi) & 0 & \sin(\psi) \\ 0 & 1 & 0 \\ -\sin(\psi) & 0 & \cos(\psi) \end{pmatrix}$.

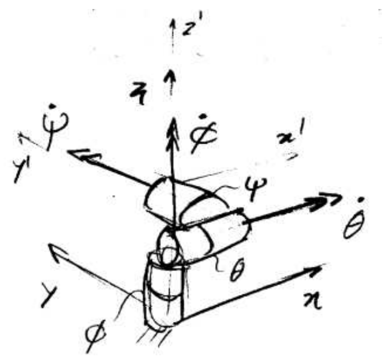

Figure 10.6: Definition of the rotation matrix and the angular velocities

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Now the angular velocities in terms of the Euler angles $\mathbf{p}$ and their time derivatives $\dot{\mathbf{p}}$. First for the space fixed angular velocities,

$$\boldsymbol{\omega} = \mathbf{A}(\mathbf{p})\,\dot{\mathbf{p}}$$

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} + \mathbf{R}_\phi \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\phi \mathbf{R}_\theta \begin{pmatrix} 0 \\ \dot{\psi} \\ 0 \end{pmatrix}$$

$$\begin{array}{ccccc} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} & = & \begin{pmatrix} 0 & \cos(\phi) & -\sin(\phi)\cos(\theta) \\ 0 & \sin(\phi) & \cos(\phi)\cos(\theta) \\ 1 & 0 & \sin(\theta) \end{pmatrix} & \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \\ \boldsymbol{\omega} & = & \mathbf{A}(\mathbf{p}) & \dot{\mathbf{p}} \end{array}$$

Next the body fixed angular velocities,

$$\boldsymbol{\omega}' = \mathbf{A}'(\mathbf{p})\,\dot{\mathbf{p}}$$

$$\boldsymbol{\omega}' = \begin{pmatrix} 0 \\ \dot{\psi} \\ 0 \end{pmatrix} + \mathbf{R}_\psi^T \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\psi^T \mathbf{R}_\theta^T \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix}$$

$$\begin{array}{ccccc} \begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} & = & \begin{pmatrix} -\sin(\psi)\cos(\theta) & \cos(\psi) & 0 \\ \sin(\theta) & 0 & 1 \\ \cos(\psi)\cos(\theta) & \sin(\psi) & 0 \end{pmatrix} & \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \\ \boldsymbol{\omega}' & = & \mathbf{A}'(\mathbf{p}) & \dot{\mathbf{p}} \end{array}$$

And for the space fixed angular velocities the inverse,

$$\begin{array}{ccccc} \dot{\mathbf{p}} & = & & \left(\mathbf{A}'\left(\mathbf{p}\right)\right)^{-1} & \boldsymbol{\omega}' \\ \left(\begin{array}{c} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{array}\right) & = & \frac{1}{\cos\left(\theta\right)} & \left(\begin{array}{ccc} -\sin\left(\psi\right) & 0 & \cos\left(\psi\right) \\ \cos\left(\theta\right)\cos\left(\psi\right) & 0 & \cos\left(\theta\right)\sin\left(\psi\right) \\ \sin\left(\theta\right)\sin\left(\psi\right) & \cos\left(\theta\right) & -\sin\left(\theta\right)\cos\left(\psi\right) \end{array}\right) & \left(\begin{array}{c} \omega'_x \\ \omega'_y \\ \omega'_z \end{array}\right) \end{array}$$

Note the singularity at $\theta = \pi/2 \pm k\pi$!

Recall that the state of the system is described by the coodinates of the cm, $\mathbf{x}_{G_i}$, the Euler angles, $\mathbf{p}_i$, the velocities of the cm of the bodies, $\mathbf{v}_{G_i}$, and the angular velocities of the bodies expressed in the body fixed frame, $\boldsymbol{\omega}'_i$.
Then the state equations are, first for the coordinates and Euler angles,

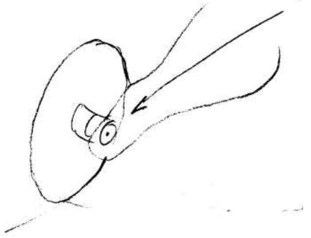
$$\left.\begin{array}{ccc} \dot{\mathbf{x}}_{G_i} & = & \mathbf{v}_{G_i} \\ \dot{\mathbf{p}}_i & = & \left(\mathbf{A}'\left(\mathbf{p}_i\right)\right)^{-1}\boldsymbol{\omega}'_i \end{array}\right\} \text{for each body } i = 1\cdots 4 \tag{10.1}$$

and next for the velocities, that is the constrained Equations of Motion for the complete system,

$$\left[\begin{array}{cccccc|c} m_1 I_3 & & & & & & \\ & J'_{G_1} & & & & & \\ & & m_2 I_3 & & & & \mathbf{D}^T \\ & & & J'_{G_2} & & & \\ & & & & \ddots & & \\ \hline & & & & & & \\ & & & & & & \\ & & \mathbf{D} & & & & \\ & & & & & & \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{v}}_{G_1} \\ \dot{\boldsymbol{\omega}}'_1 \\ \dot{\mathbf{v}}_{G_2} \\ \dot{\boldsymbol{\omega}}'_2 \\ \vdots \\ \hline \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{21} \end{array}\right] = \left[\begin{array}{l} \Sigma\mathbf{f}_1 \\ \Sigma\mathbf{M}'_1 - \boldsymbol{\omega}'_1 \times \left(J'_{G_1}\boldsymbol{\omega}'_1\right) \\ \Sigma\mathbf{f}_2 \\ \Sigma\mathbf{M}'_2 - \boldsymbol{\omega}'_2 \times \left(J'_{G_2}\boldsymbol{\omega}'_2\right) \\ \vdots \\ \hline \vdots \\ \vdots \\ \mathbf{g}_i \\ \vdots \end{array}\right] \tag{10.2}$$

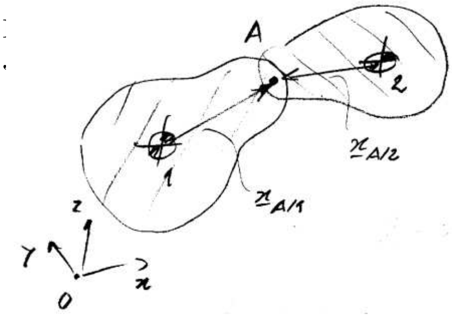
where $\mathbf{g}_i = -\mathbf{D}^2\left(\mathbf{v}_{G_i}, \boldsymbol{\omega}'_i\right)\left(\mathbf{v}_{G_i}, \boldsymbol{\omega}'_i\right)$” are the convective terms.
The matrix $\mathbf{D}$ is the Jacobian $\mathbf{D}_{h,j}$ from the holonomic constraints $\boldsymbol{\epsilon}_h = \mathbf{D}_h\left(\mathbf{x}_{G_j}, \mathbf{p}_j\right)$, and the Jacobian from the additional nonholonomic constraints, $\dot{\boldsymbol{\epsilon}}_{nh} = \mathbf{D}_{nh}\left(\dot{\mathbf{x}}_{G_j}, \dot{\mathbf{p}}_j\right)$, which are...... wait and see!

10.2 Constraints



Revolute Joint =
2 Points coincide → 3
2 hinge axes parallel → 2

Total # of constraints → 5



First look at two points coincide:

$$\begin{aligned}\boldsymbol{\epsilon} &= \mathbf{x}_{A_2} - \mathbf{x}_{A_1} = \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{pmatrix} \\ \boldsymbol{\epsilon} &= \mathbf{x}_2 + \mathbf{x}_{A/2} - \mathbf{x}_1 - \mathbf{x}_{A/1} \\ \boldsymbol{\epsilon} &= \mathbf{x}_2 + \mathbf{R}_2\mathbf{x}'_{A/2} - \mathbf{x}_1 - \mathbf{R}_1\mathbf{x}'_{A/1}\end{aligned}$$

with $\mathbf{R}_1 = \mathbf{R}(\mathbf{p}_1)$, $\mathbf{R}_2 = \mathbf{R}(\mathbf{p}_2)$ and where $\mathbf{x}'_{A/2}, \mathbf{x}'_{A/1}$ are constant vectors.

Figure 10.8: Two Points Coincide

Velocities:

$$\begin{aligned}\dot{\boldsymbol{\epsilon}} &= \dot{\mathbf{x}}_2 + \boldsymbol{\omega}_2 \times \mathbf{x}_{A/2} - \dot{\mathbf{x}}_1 - \boldsymbol{\omega}_1 \times \mathbf{x}_{A/1} \\ &= \mathbf{v}_2 + \boldsymbol{\omega}_2 \times \mathbf{x}_{A/2} - \mathbf{v}_1 - \boldsymbol{\omega}_1 \times \mathbf{x}_{A/1}\end{aligned}$$

Accelerations:

$$\begin{aligned}\ddot{\boldsymbol{\epsilon}} = \quad & \dot{\mathbf{v}}_2 + \dot{\boldsymbol{\omega}}_2 \times \mathbf{x}_{A/2} - \dot{\mathbf{v}}_1 - \dot{\boldsymbol{\omega}}_1 \times \mathbf{x}_{A/1} \\ & + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{x}_{A/2}) - \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{x}_{A/1})\end{aligned}$$

These last equations are the constraint equations we have to add to the equations of motion but .... we work in a body fixed $\boldsymbol{\omega}'$ instead of space $\boldsymbol{\omega}$.

Velocities: $\dot{\boldsymbol{\epsilon}} = \mathbf{v}_2 + \mathbf{R}_2 \left(\boldsymbol{\omega}'_2 \times \mathbf{x}'_{A/2}\right) - \mathbf{v}_1 - \mathbf{R}_1 \left(\boldsymbol{\omega}'_1 \times \mathbf{x}'_{A/1}\right)$

Which we can write in a matrix vector expression by using the tilde matrix for the cross product.

$$\dot{\boldsymbol{\epsilon}} = \mathbf{v}_2 - \mathbf{R}_2\tilde{\mathbf{x}}'_{A/2}\boldsymbol{\omega}'_2 - \mathbf{v}_1 + \mathbf{R}_1\tilde{\mathbf{x}}'_{A/1}\boldsymbol{\omega}'_1$$

or in matrix-vector form:

$$\dot{\boldsymbol{\epsilon}} = \begin{bmatrix} -\mathbf{I} & \mathbf{R}_1\tilde{\mathbf{x}}'_{A/1} & \mathbf{I} & -\mathbf{R}_2\tilde{\mathbf{x}}'_{A/2} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \boldsymbol{\omega}'_1 \\ \mathbf{v}_2 \\ \boldsymbol{\omega}'_2 \end{bmatrix}$$

The matrix $\mathbf{R}_1\tilde{\mathbf{x}}'_{A/1}$ is the Jacobian of the constraints, "$\mathbf{D}$". The constraint on the accelerations is now:

$$\ddot{\boldsymbol{\epsilon}} = \begin{bmatrix} -\mathbf{I} & \mathbf{R}_1\tilde{\mathbf{x}}'_{A/1} & \mathbf{I} & -\mathbf{R}_2\tilde{\mathbf{x}}'_{A/2} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}}_1 \\ \dot{\boldsymbol{\omega}}'_1 \\ \dot{\mathbf{v}}_2 \\ \dot{\boldsymbol{\omega}}'_2 \end{bmatrix} + \mathbf{g}$$

where the convective terms $\mathbf{g}$ can be calculated from the state as:

$$\mathbf{g} = \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{x}_{A/2}) - \boldsymbol{\omega}_1 \times (\boldsymbol{\omega}_1 \times \mathbf{x}_{A/1})$$

Next look at the two hinge axes being parallel constraints. We want the two hinge axis $\hat{\mathbf{e}}_{x_1}$ and $\hat{\mathbf{e}}_{x_2}$ to be parallel or in other words: $\hat{\mathbf{e}}_{x_2} \perp \hat{\mathbf{e}}_{y_1}$ and $\hat{\mathbf{e}}_{x_2} \perp \hat{\mathbf{e}}_{z_1}$.

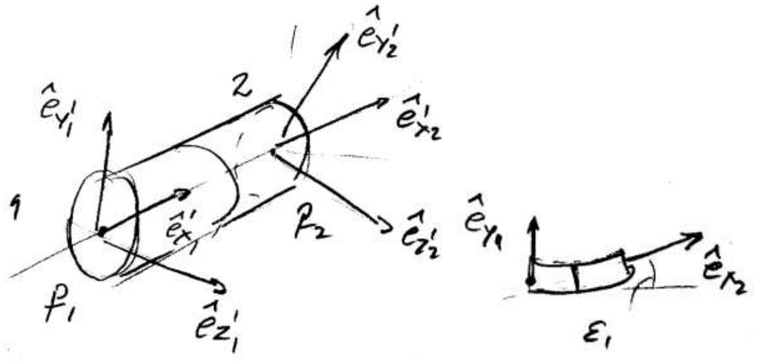


Figure 10.9: Hinge axis being parallel (left), and a bending mode of the hinge (right).

$$\begin{aligned} \epsilon_1 &= \hat{\mathbf{e}}_{y_1}^T\hat{\mathbf{e}}_{x_2} \quad (\text{ like bending about } \hat{\mathbf{e}}_{z_1}) \\ \epsilon_2 &= \hat{\mathbf{e}}_{z_1}^T\hat{\mathbf{e}}_{x_2} \quad (\text{ like bending about } \hat{\mathbf{e}}_{y_1}) \end{aligned}$$

or with $\mathbf{R}_1 = \mathbf{R}(\mathbf{p}_1)$ and $\mathbf{R}_2 = \mathbf{R}(\mathbf{p}_2)$ and referring to the body fixed prime vectors which are constant:

$$\begin{aligned} \epsilon_1 &= \left(\mathbf{R}_1\hat{\mathbf{e}}'_{y_1}\right)^T \left(\mathbf{R}_2\hat{\mathbf{e}}'_{x_2}\right) &= \hat{\mathbf{e}}'^T_{y_1}\mathbf{R}_1^T\mathbf{R}_2\hat{\mathbf{e}}'_{x_2} \\ \epsilon_2 &= \left(\mathbf{R}_1\hat{\mathbf{e}}'_{z_1}\right)^T \left(\mathbf{R}_2\hat{\mathbf{e}}'_{x_2}\right) &= \hat{\mathbf{e}}'^T_{z_1}\mathbf{R}_1^T\mathbf{R}_2\hat{\mathbf{e}}'_{x_2} \end{aligned}$$

Velocities: here we use the rule that the time derivative of a unit vector is omega cross this unit vector, as in $\dot{\hat{\mathbf{e}}}_i = \boldsymbol{\omega}_i \times \hat{\mathbf{e}}_i$, making

$$\begin{aligned} \dot{\epsilon}_1 \quad &= \dot{\hat{\mathbf{e}}}_{y_1}^T\hat{\mathbf{e}}_{x_2} + \hat{\mathbf{e}}_{y_1}^T\dot{\hat{\mathbf{e}}}_{x_2} \\ &= \left(\boldsymbol{\omega}_1 \times \hat{\mathbf{e}}_{y_1}\right)^T \hat{\mathbf{e}}_{x_2} + \hat{\mathbf{e}}_{y_1}^T\left(\boldsymbol{\omega}_2 \times \hat{\mathbf{e}}_{x_2}\right) \end{aligned}$$

But we prefer the body fixed coordinate system with $\boldsymbol{\omega}'$.

$$\begin{aligned} \dot{\epsilon}_1 \quad &= \left(\mathbf{R}_1\left(\boldsymbol{\omega}'_1 \times \hat{\mathbf{e}}'_{y_1}\right)\right)^T \hat{\mathbf{e}}_{x_2} + \hat{\mathbf{e}}_{y_1}^T\left(\mathbf{R}_2\left(\boldsymbol{\omega}'_2 \times \hat{\mathbf{e}}'_{x_2}\right)\right) \\ \dot{\epsilon}_1 \quad &= -\hat{\mathbf{e}}_{x_2}^T\mathbf{R}_1\tilde{\hat{\mathbf{e}}}'_{y_1}\boldsymbol{\omega}'_1 - \hat{\mathbf{e}}_{y_1}^T\mathbf{R}_2\tilde{\hat{\mathbf{e}}}'_{x_2}\boldsymbol{\omega}'_2 \end{aligned}$$

Apparently the Jacobian matrix $\mathbf{D}$ is:

$$\frac{\partial \dot{\epsilon}_1}{\partial \boldsymbol{\omega}_1'} = -\hat{\mathbf{e}}_{x_2}^T \mathbf{R}_1 \tilde{\hat{\mathbf{e}}}_{y_1}' \text{ and } \frac{\partial \dot{\epsilon}_1}{\partial \boldsymbol{\omega}_2'} = -\hat{\mathbf{e}}_{y_1}^T \mathbf{R}_2 \tilde{\hat{\mathbf{e}}}_{x_2}'$$

Accelerations:

$$\begin{aligned} \ddot{\epsilon}_1 = \;& \left(\dot{\boldsymbol{\omega}}_1 \times \hat{\mathbf{e}}_{y_1}\right)^T \hat{\mathbf{e}}_{x_2} + \hat{\mathbf{e}}_{y_1}^T \left(\dot{\boldsymbol{\omega}}_2 \times \hat{\mathbf{e}}_{x_2}\right) + \\ & \left(\boldsymbol{\omega}_1 \times \left(\boldsymbol{\omega}_1 \times \hat{\mathbf{e}}_{y_1}\right)\right)^T \hat{\mathbf{e}}_{x_2} + \left(\boldsymbol{\omega}_1 \times \hat{\mathbf{e}}_{y_1}\right)^T \left(\boldsymbol{\omega}_2 \times \hat{\mathbf{e}}_{x_2}\right) + \\ & \left(\boldsymbol{\omega}_1 \times \hat{\mathbf{e}}_{y_1}\right)^T \left(\boldsymbol{\omega}_2 \times \hat{\mathbf{e}}_{x_2}\right) + \hat{\mathbf{e}}_{y_1}^T \left(\boldsymbol{\omega}_2 \times \left(\boldsymbol{\omega}_2 \times \hat{\mathbf{e}}_{x_2}\right)\right) \end{aligned}$$

Be aware that the convective term $\mathbf{g}_1$ goes to the right-hand side of the Equations of motion.

$$\begin{aligned} \mathbf{g}_1 = \;& \left(\boldsymbol{\omega}_1 \times \left(\boldsymbol{\omega}_1 \times \hat{\mathbf{e}}_{y_1}\right)\right)^T \hat{\mathbf{e}}_{x_2} + \left(\boldsymbol{\omega}_1 \times \hat{\mathbf{e}}_{y_1}\right)^T \left(\boldsymbol{\omega}_2 \times \hat{\mathbf{e}}_{x_2}\right) + \\ & \left(\boldsymbol{\omega}_1 \times \hat{\mathbf{e}}_{y_1}\right)^T \left(\boldsymbol{\omega}_2 \times \hat{\mathbf{e}}_{x_2}\right) + \hat{\mathbf{e}}_{y_1}^T \left(\boldsymbol{\omega}_2 \times \left(\boldsymbol{\omega}_2 \times \hat{\mathbf{e}}_{x_2}\right)\right) \end{aligned}$$

and of course

$$\frac{\partial \ddot{\epsilon}_1}{\partial \dot{\boldsymbol{\omega}}_1} = \frac{\partial \dot{\epsilon}_1}{\partial \boldsymbol{\omega}_1}$$

so this Jacobian of the acceleration constraint is the same as the one for the velocities constraint, always!

Wheel contact constraints

The velocity of contact point c is

$$\dot{\mathbf{x}}_c = \dot{\mathbf{x}}_G + \boldsymbol{\omega} \times \mathbf{r}$$

No slip means: $\dot{\mathbf{x}}_c = \mathbf{0}$. Actually we have one holonomic contact condition

$$z_G = r\cos(\theta),$$

and two nonholonomic or no-slip conditions

$$\begin{aligned} \dot{x}_c &= 0 \\ \dot{y}_c &= 0 \end{aligned} \; .$$

But ... I like to think in a moving coordinate system.

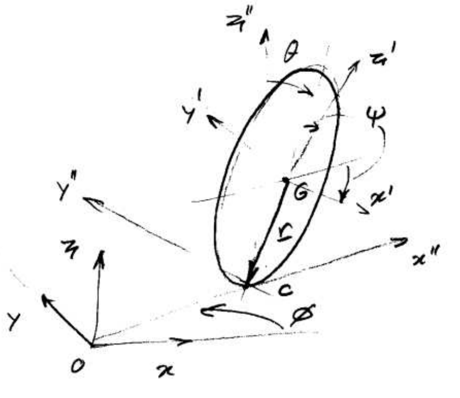


Figure 10.10: Definition coordinate systems for the wheel contact constraint

$$\begin{aligned} \dot{x}_c^{''} &\neq 0 && \text{zero longitudinal slip} \\ \dot{y}_c^{''} &\neq 0 && \text{zero lateral slip} \end{aligned}$$

$$\dot{\mathbf{x}}_c^{''} = \dot{\mathbf{x}}_G^{''} + \boldsymbol{\omega}^{''} \times \mathbf{r}^{''}$$

Now with

$$\dot{\mathbf{x}}_G^{''} = \mathbf{R}_\phi^T \dot{\mathbf{x}}_G$$

and

$$\left(\boldsymbol{\omega}^{''} \times \mathbf{r}^{''}\right) = \mathbf{R}_\theta \mathbf{R}_\psi \left(\boldsymbol{\omega}' \times \mathbf{r}'\right).$$

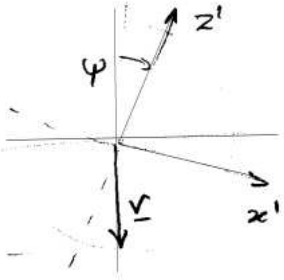


Figure 10.11: r,r'

and the radius vector in the local reference frame, see Figure 10.11,

$$\mathbf{r}' = \begin{pmatrix} r\sin(\psi) \\ 0 \\ -r\cos(\psi) \end{pmatrix}$$

we can evaluate $\dot{\mathbf{x}}_c''$ (which is tedious so use Matlab symbolic toolbox or Maple!). We end up with the linear equations for the velocities of the contact point c in terms of the coordinates and velocities of the wheel,

$$\begin{bmatrix} \dot{x}'' \\ \dot{y}'' \\ \dot{z}'' \end{bmatrix}_c = \left[\begin{array}{cccccc} \dot{x} & \dot{y} & \dot{z} & \omega_x' & \omega_y' & \omega_z' \\ \cos(\phi) & \sin(\phi) & 0 & 0 & -r & 0 \\ -\sin(\phi) & \cos(\phi) & 0 & r\cos(\theta)\cos(\psi) & 0 & r\cos(\theta)\sin(\psi) \\ 0 & 0 & 1 & r\sin(\theta)\cos(\psi) & 0 & r\sin(\theta)\sin(\psi) \end{array}\right]$$

The last entry, $\dot{z}''$, is the holonomic constraint $\dot{z}'' = \dot{z} + r\sin(\theta)\dot{\theta} = 0$ (Check this with $\dot{\theta}$ from $\dot{p} = (\mathbf{A}'(\mathbf{p}))^{-1}\boldsymbol{\omega}'$ in terms of $\boldsymbol{\omega}'$ i.e. $\dot{\theta} = \cos(\psi)\omega_x' + \sin(\psi)\omega_z'$), which comes from the holonomic contact condition,

$$\epsilon_1 = z - r\cos(\theta).$$

The first two $\dot{x}''$ and $\dot{y}''$ are the two nonholonomic constraints, or no-slip conditions, in the longitudinal and lateral direction of the wheel,

$$\begin{array}{ll} \dot{\epsilon}_2 = \cos(\phi)\dot{x} + \sin(\phi)\dot{y} - r\omega_y' & (\text{ longitudinal slip }) \\ \dot{\epsilon}_3 = -\sin(\phi)\dot{x} + \cos(\phi)\dot{y} + r\omega_x'\cos(\theta)\cos(\psi) + r\omega_z'\cos(\theta)\sin(\psi) & (\text{ lateral slip }) \end{array}$$

Now we only have to find the convective terms for the wheel constraint. First substitute $\boldsymbol{\omega}'$ in terms of $\mathbf{p}$ and $\dot{\mathbf{p}}$ in the $\dot{\boldsymbol{\epsilon}}$ equations.

$$\begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \end{bmatrix} = \left[\begin{array}{cccccc} \dot{x} & \dot{y} & \dot{z} & \dot{\phi} & \dot{\theta} & \dot{\psi} \\ 0 & 0 & 1 & 0 & r\sin(\theta) & 0 \\ \cos(\phi) & \sin(\phi) & 0 & -r\sin(\theta) & 0 & -r \\ -\sin(\phi) & \cos(\phi) & 0 & 0 & r\cos(\theta) & 0 \end{array}\right]$$

Next differentiate with respect to time, and forget all $\ddot{\ }$ terms, this gives us the convective terms,

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} r\cos(\theta)\dot{\theta}^2 \\ -\sin(\phi)\dot{x}\dot{\phi} + \cos(\phi)\dot{y}\dot{\phi} - r\cos(\theta)\dot{\phi}\dot{\theta} \\ -\cos(\phi)\dot{x}\dot{\phi} - \sin(\phi)\dot{y}\dot{\phi} - r\sin(\theta)\dot{\theta}^2 \end{bmatrix}$$

Finally we can use numerical integration on the accelerations from the equations of motion and on the velocities to find the motion in time. You can use any method, but I prefer RK4 with a fixed stepsize. This is a fast and accurate method. The only problem is to find a reasonable stepsize which gives fast and accurate results. Start with $h = 0.01$ sec and decrease or increase according to the results. Use the Coordinate Projection Method to minimize the constraint errors after each numerical integration step. In this scheme we use the Moore-Penrose pseudo-inverse $\mathbf{D}^+$ of the Jacobian of the constraints $\mathbf{D}$, as in $\mathbf{D}^+ = \mathbf{D}^T \left(\mathbf{D}\mathbf{D}^T\right)^{-1}$. Where the Jacobian $\mathbf{D} = \frac{\partial \boldsymbol{\epsilon}}{\partial(\mathbf{x}_i, \mathbf{p}_i)}$, is the partial derivatives of the constraints with respect to the state variables $\mathbf{x}_i$ and $\mathbf{p}_i$. This is not equal to the $\mathbf{D}$ which we have used in the constraint equations of motion, since there we had the partial derivatives of the velocities with respect to the state variable $\mathbf{v}_i$ and $\boldsymbol{\omega}'_i$ as in $\mathbf{D} = \frac{\partial \dot{\boldsymbol{\epsilon}}}{\partial\left(\mathbf{v}_i, \boldsymbol{\omega}'_i\right)}$. But given the latter we can easily transform to the former with the matrices $\left(\mathbf{A}'\left(\mathbf{p}_i\right)\right)$ since $\frac{\partial \boldsymbol{\omega}'_i}{\partial \dot{\mathbf{p}}_i} = \left(\mathbf{A}'\left(\mathbf{p}_i\right)\right)$ and thus get,

$$\frac{\partial \boldsymbol{\epsilon}}{\partial\left(\mathbf{x}_i, \mathbf{p}_i\right)} = \frac{\partial \dot{\boldsymbol{\epsilon}}}{\partial\left(\dot{\mathbf{x}}_i, \dot{\mathbf{p}}_i\right)} = \left[\frac{\partial \dot{\boldsymbol{\epsilon}}}{\partial \mathbf{v}_i}, \frac{\partial \dot{\boldsymbol{\epsilon}}}{\partial \boldsymbol{\omega}'_i} \mathbf{A}'\left(\mathbf{p}_i\right)\right]$$

As an example the "two points coincide" constraint, for the velocity constraint equations we derived

$$\dot{\boldsymbol{\epsilon}} = \begin{bmatrix} -\mathbf{I} & \mathbf{R}_1 \tilde{\mathbf{x}}'_{A/1} & \mathbf{I} & -\mathbf{R}_2 \tilde{\mathbf{x}}'_{A/2} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \boldsymbol{\omega}'_1 \\ \mathbf{v}_2 \\ \boldsymbol{\omega}'_2 \end{bmatrix}.$$

which then transforms to small changes in the coordinates and Euler angles as

$$\Delta \boldsymbol{\epsilon} = \begin{bmatrix} -\mathbf{I} & \mathbf{R}_1 \tilde{\mathbf{x}}'_{A/1}\left(\mathbf{A}'_1\right) & \mathbf{I} & -\mathbf{R}_2 \tilde{\mathbf{x}}'_{A/2}\left(\mathbf{A}'_2\right) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{x}_1 \\ \Delta \mathbf{p}_1 \\ \Delta \mathbf{x}_2 \\ \Delta \mathbf{p}_2 \end{bmatrix}$$

Note that this Coordinate Projection Method only operates on the holonomic constraints, in our case the 17 constraints.

Finally we have to find speeds, $\mathbf{v}_i$ and $\boldsymbol{\omega}'_i$ which fulfil the velocity constraints. So now we include our non-holonomic constraints. To minimize the velocity constraint errors we use the Coordinate Projection Method again but since we work in this velocity space we can use the Jacobian $\mathbf{D}$ as derived for the constrained equations of motion. Since the equations are linear in the speeds this is a one step iteration!

SUCCES!