# Lecture Notes
# Multibody Dynamics B, wb1413


A. L. Schwab & Guido M.J. Delhaes

Laboratory for Engineering Mechanics
Mechanical Engineering
Delft University of Technolgy
The Netherlands


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# Chapter 9

# Rotation Matrix, Euler Angles & Euler Parameters

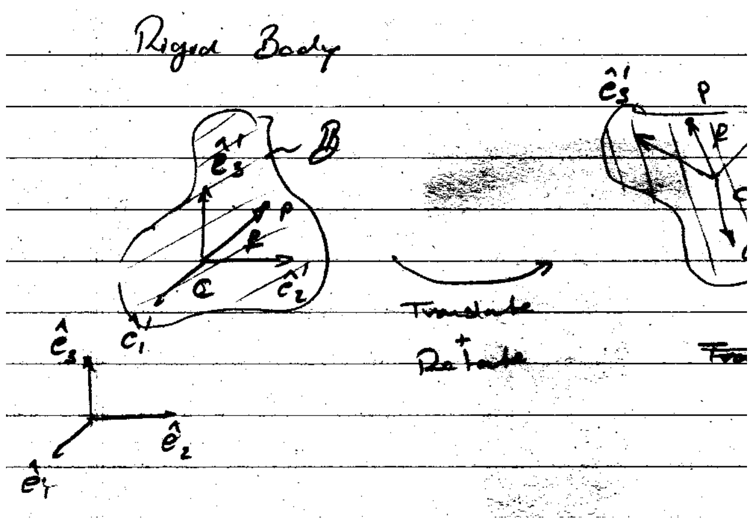


Figure 9.1: Rigid Body coordinate system

The motion of a rigid body in space can be described by the sum of a translation and a rotation. For instance the translation of a point $C$ of the body, and for the rotation we look at the change in orientation of a body fixed coordinate system described by the motion of the body fixed base vectors $\mathbf{e}'_i$, where $i = 1 \cdots 3$.

Here we focus on the rotation so lets forget the translational part. We follow the position of an arbitrary point $P$, which is fixed in rigid body $B$, during rotation.

After rotation, point $P$ is

$$\mathbf{p} = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3$$

with $p_i$ the algebraic components in the space fixed coordinate system, and $\mathbf{e}_i$ with $i = 1 \cdots 3$ the space fixed coordinate system. Or to be more specific, the algebraic components are the numbers you put into Matlab when you write a vector like:

```
> v =[1;6;-3];
```

And the $\mathbf{e}_i$ are just a notation for the vector base. Back to point $P$, we could also write $\mathbf{p}$ in terms of the body fixed coordinate system $\mathbf{e}'_i$, as in

$$\mathbf{p} = p'_1\mathbf{e}'_1 + p'_2\mathbf{e}'_2 + p'_3\mathbf{e}'_3$$

with $p'_i$ the algebraic components in the body fixed coordinate system. Now if $B$ is a rigid body, then the algebraic components $p'_i$ are constant!

How do we find the components $p_i$? If the vector base is an orthogonal base then the dot-products of the base vectors is $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta, with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. By taking the dot product of $\mathbf{p}$ and a base vector $\mathbf{e}_i$ we get the $i^{\text{th}}$ algebraic component, as in

$$p_i = \mathbf{p} \cdot \mathbf{e}_i$$

But if we use the expression for $\mathbf{p}$ in the body fixed base $\mathbf{e}'_i$ we get

$$p_i = (p'_1\mathbf{e}'_1 + p'_2\mathbf{e}'_2 + p'_3\mathbf{e}'_3) \cdot \mathbf{e}_i$$

Or expanded for all indices and written out in matrix vector form

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_1 \cdot \mathbf{e}_1 & \mathbf{e}'_2 \cdot \mathbf{e}_1 & \mathbf{e}'_3 \cdot \mathbf{e}_1 \\ \mathbf{e}'_1 \cdot \mathbf{e}_2 & \mathbf{e}'_2 \cdot \mathbf{e}_2 & \mathbf{e}'_3 \cdot \mathbf{e}_2 \\ \mathbf{e}'_1 \cdot \mathbf{e}_3 & \mathbf{e}'_2 \cdot \mathbf{e}_3 & \mathbf{e}'_3 \cdot \mathbf{e}_3 \end{pmatrix} \begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix}$$

Or, in other words, given the body fixed components $p'_i$ of point $P$ we can find the space fixed components $p_i$ from

$$p_i = \left(\mathbf{e}_i \cdot \mathbf{e}'_j\right) p'_j \text{ with } i, j = 1 \cdots 3$$

The 9 dot-products of the 2 times 3 base vectors is what we call the Rotation matrix $\mathbf{R} = R_{ij}$. This rotation matrix transforms the algebraic components from the body fixed frame to the algebraic components of the space fixed frame, as in

$$p_i = R_{ij}p'_j \text{ with } R_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j \text{ where } i, j = 1 \cdots 3.$$

The inverse of $R_{ij}$ follows directly from:

$$\begin{aligned} p'_i &= \mathbf{p} \cdot \mathbf{e}'_i = p_j\mathbf{e}_j \cdot \mathbf{e}'_i \\ p'_i &= R^{-1}_{ij} p_j \text{ with } R^{-1}_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j \end{aligned}$$

Apparently $R^{-1}_{ij} = R_{ji}$ or in other words, the inverse is the transpose as in $\mathbf{R}^{-1} = \mathbf{R}^T$. Which we can also write as $\mathbf{R}\mathbf{R}^T = \mathbf{I}$. A matrix which fulfil this requirement is called an orthogonal matrix. Such a matrix has all eigenvalues $\lambda \pm 1$.

Now $\mathbf{R}$ has 9 components and $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ imposes 6 independent conditions on $\mathbf{R}$. (And not 9 because of the symmetry $\left(\mathbf{R}\mathbf{R}^T\right)^T = \mathbf{R}\mathbf{R}^T$. So you could parameterize $\mathbf{R}$ with $9 - 6 = 3$ independent parameters.

Let's check this in 2D.

| | | |
|---|---|---|
| $\mathbf{R}$ | 4 | components |
| $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ | 3 | conditions |
| | 1 | parameters to describe $\mathbf{R}$ |

When we continue, then in a 4 dimensional space (I have no clear concept of such a space, but anyway just for example's sake), we can parameterize this $4 \times 4$ rotation matrix with $16 - 10 = 6$ independent parameters!

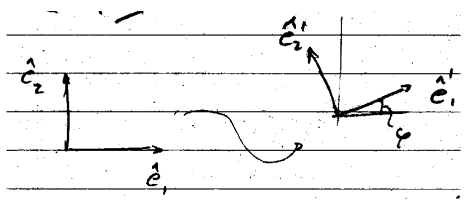

Figure 9.2: Fixed and rotated coordinate system

The rotation matrix in 2D can be derived by looking at Figure 9.2 and first writing down the base vectors,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{e}'_1 = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} \mathbf{e}'_2 = \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \end{pmatrix}$$

then

$$R_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \text{ and } R^{-1}_{ij} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

## 9.1 Euler Angles

One way to parameterize the 3D rotation matrix $\mathbf{R}$ is by Euler angles. Euler angles is a kind of recipe which goes like this:

Start with the space fixed coordinate system and align the body fixed coordinate system with the space fixed system.

1. Rotate with an angle $\phi$ (phi) about the $z$-axis.
2. Rotate with an angle $\theta$ (theta) about the rotated $x$-axis.
3. Rotate with an angle $\psi$ (psi) about the rotated $z$-axis.

Sometimes these Euler angles are called $z$–$x$–$z$ or 3–1–3 angles or precession, rotation and spin angles.

How does the rotation matrix $\mathbf{R}$ then look? Well first of all the rotation matrix $\mathbf{R}$ is defined as the transformation of the body fixed coordinates $\mathbf{x}'$ into the space fixed coordinates $\mathbf{x}$ as in,

$$\mathbf{x} = \mathbf{R}\mathbf{x}'$$

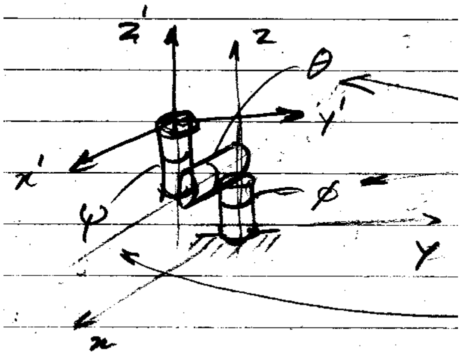


Figure 9.3: Euler angles $z$–$x$–$z$ denoted by $\phi, \theta, \psi$ depicted by the so-called "cans in series" to show the order of rotation.

We can derive this rotation matrix by divide and conquer, i.e. looking at every individual transformation first and then combining the three, see Figure 9.4.

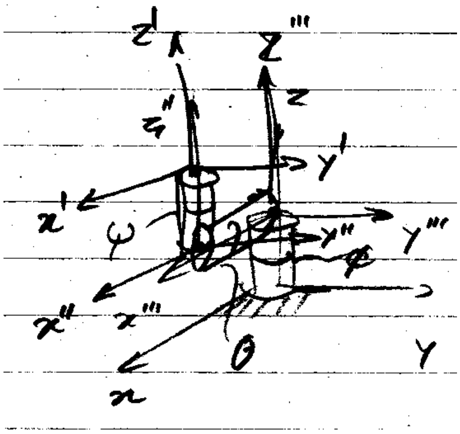


Figure 9.4: Euler angles $z$–$x$–$z$ depicted by cans in series, together with the three individual rotations $\phi, \theta, \psi$ and corresponding $'''$, $''$ and $'$ coordinate systems.

$$\mathbf{x} = \mathbf{R}_\phi \mathbf{x}''' \quad \mathbf{R}_\phi = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\mathbf{x}''' = \mathbf{R}_\theta \mathbf{x}'' \quad \mathbf{R}_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$
$$\mathbf{x}'' = \mathbf{R}_\psi \mathbf{x}' \quad \mathbf{R}_\psi = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Putting everything together:

$$\mathbf{x} = \mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi \mathbf{x}' = \mathbf{R}\mathbf{x}'$$

$$
\begin{aligned}
\mathbf{R} &= \mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi \\
&= \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \cos(\theta)\sin(\psi) & \cos(\theta)\cos(\psi) & -\sin(\theta) \\ \sin(\theta)\sin(\psi) & \sin(\theta)\cos(\psi) & \cos(\theta) \end{pmatrix}
\end{aligned}
$$

The rotation matrix $\mathbf{R}$ expressed in the $z$–$x$–$z$ Euler angles $\phi, \theta, \psi$ is now,

$$
\mathbf{R} = \begin{pmatrix} \cos(\phi)\cos(\psi) - \sin(\phi)\cos(\theta)\sin(\psi) & -\cos(\phi)\sin(\psi) - \sin(\phi)\cos(\theta)\cos(\psi) & \sin(\phi)\sin(\theta) \\ \sin(\phi)\cos(\psi) + \cos(\phi)\cos(\theta)\sin(\psi) & -\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\theta)\cos(\psi) & -\cos(\phi)\sin(\theta) \\ \sin(\theta)\sin(\psi) & \sin(\theta)\cos(\psi) & \cos(\theta) \end{pmatrix}
$$

Now $\mathbf{R}_\phi \mathbf{R}_\theta \mathbf{R}_\psi \neq \mathbf{R}_\theta \mathbf{R}_\phi \mathbf{R}_\psi$ so the order of rotation is important. This means that the Euler angles $(\phi, \theta, \psi)$ are *no* vectors! Because, if Euler angles were vectors then the order would not matter, but it does!

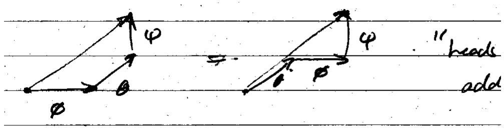


Figure 9.5: If Euler angles were vectors the order would not matter as pictured here in the "head to head" addition.

## 9.2 Euler angles and angular velocities

The next step in describing orientation is space is to look at the change with time, angular velocities. Recall that, after translation and rotation, we can find

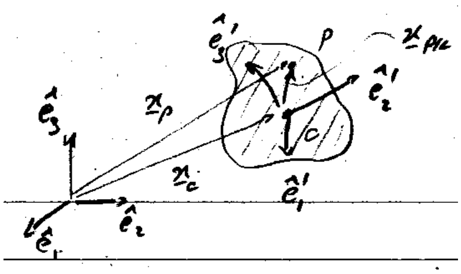


Figure 9.6: Sketch fixed and body fixed coordinate system

any point $p$ in the body from,

$$
\begin{aligned}
\mathbf{x}_p &= \mathbf{x}_c + \mathbf{x}_{p/c} \\
&= \mathbf{x}_c + \mathbf{R}\mathbf{x}'_{p/c} \text{ with } \mathbf{R} = \mathbf{R}(\phi, \theta, \psi)
\end{aligned}
$$

,where the body fixed coordinates $\mathbf{x}'_{p/c}$ are constant for a rigid body.

We now find the velocities by differentatin with respect to time, as in,

$$\begin{aligned}\dot{\mathbf{x}}_p &= \dot{\mathbf{x}}_c + \dot{\mathbf{x}}_{p/c} \\ &= \dot{\mathbf{x}}_c + \dot{\mathbf{R}}\mathbf{x}'_{p/c} + \mathbf{R}\dot{\mathbf{x}}'_{p/c} \text{ with } \dot{\mathbf{x}}'_{p/c} = 0\end{aligned}$$

because $\mathbf{x}'_{p/c}$ is constant for a rigid body. Now forget about the translation of the center $c$ and look at velocity of $p$ relative to $c$,

$$\dot{\mathbf{x}}_p - \dot{\mathbf{x}}_c = \dot{\mathbf{x}}_{p/c} = \dot{\mathbf{R}}\mathbf{x}'_{p/c}$$

The velocity on the left-hand side $\dot{\mathbf{x}}_{p/c}$ and the coordinates on the right-hand side $\mathbf{x}'_{p/c}$ are defined in two different coordinate systems. We like to express everything in one coordinate system. Here we now now choose to express everything in the space fixed one,

$$\mathbf{x}'_{p/c} = \mathbf{R}^T\mathbf{x}_{p/c} \rightarrow \boxed{\dot{\mathbf{x}}_{p/c} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{x}_{p/c}}$$

What is this matrix $\dot{\mathbf{R}}\mathbf{R}^T$?

We can derive this with help of the identities $\mathbf{R}\mathbf{R}^T = \mathbf{I}$. First differentiate these with respect to time results in $\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0}$. Substitute $\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T$ into this and we get $\mathbf{W} + \mathbf{W}^T = \mathbf{0}$, which means that $\mathbf{W}$ must be anti-symmetric. Such an $3 \times 3$ anti-symmetric matrix has only three independent parameters and can be written as,

$$\mathbf{W} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

Why so strange? Because this is the matrix-vector form of the cross product of two vectors in 3D,

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{v} \times \mathbf{x}$$

Which we can also write by making use of the *tilde* notation to make this anti-symmetric matrix out of a vector $\mathbf{v}$ as in,

$$\tilde{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix},$$

making $\mathbf{v} \times \mathbf{x} = \tilde{\mathbf{v}}\mathbf{x}$.

Back to the velocity expressions $\dot{\mathbf{x}}_{p/c} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{x}_{p/c}$. Next we define this anti-symmetric matrix $\boxed{\dot{\mathbf{R}}\mathbf{R}^T = \tilde{\boldsymbol{\omega}}}$ with the angular velocity vector $\tilde{\boldsymbol{\omega}} = (\omega_x, \omega_y, \omega_z)$ and then rewrite the velocity expressions as,

$$\begin{aligned}\dot{\mathbf{x}}_{p/c} &= \boldsymbol{\omega} \times \mathbf{x}_{p/c} \\ &= \tilde{\boldsymbol{\omega}}\mathbf{x}_{p/c}\end{aligned}$$

Now how do we find $\boldsymbol{\omega}$ from $\dot{\mathbf{R}}\mathbf{R}^T = \tilde{\boldsymbol{\omega}}$? Just work out the time derivative $\dot{\mathbf{R}}$ by first taking the partial derivatives of $\mathbf{R}$ with respect to the Euler angles $\phi, \theta, \psi$ and then these with time, as in,

$$\frac{\partial \mathbf{R}}{\partial \phi}\mathbf{R}^T \dot{\phi} + \frac{\partial \mathbf{R}}{\partial \theta}\mathbf{R}^T \dot{\theta} + \frac{\partial \mathbf{R}}{\partial \psi}\mathbf{R}^T \dot{\psi} = \tilde{\boldsymbol{\omega}}$$

These become lengthy expressions which are error prone. Moreover, since $\tilde{\boldsymbol{\omega}}$ is anti-symmetric we do twice the amount of work necessary. Twice? Even more, because we also derive three zeros on the diagonal, which actually could be used for checking the correctness. Anyway, a short-cut is to use ones intuition.

One can add-up up the three angular velocity vectors from the three hinges or

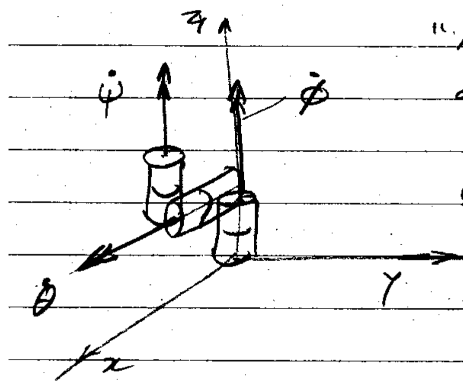


Figure 9.7: Sketch of the three velocity vectors for the three $z$–$x$–$z$ Euler angles $\phi, \theta, \psi$.

cans in series, as depicted in Figure 9.7. The three individual angular velocity vectors associated with the three hinges are,

$$\boldsymbol{\omega}_\phi = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} \quad \boldsymbol{\omega}_\theta = \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} \quad \text{and } \boldsymbol{\omega}_\psi = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} .$$

Although Euler angles are *not* vectors, these angular speeds are vectors. But note that $\boldsymbol{\omega}_\theta$ and $\boldsymbol{\omega}_\psi$ are not expressed in the global coordinate system. Therefore we have to transform them accordingly before we can add them up, resulting in

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\phi + \mathbf{R}_\phi \boldsymbol{\omega}_\theta + \mathbf{R}_\phi \mathbf{R}_\theta \boldsymbol{\omega}_\psi \Rightarrow$$

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} + \mathbf{R}_\phi \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\phi \mathbf{R}_\theta \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 & \cos(\phi) & \sin(\phi)\sin(\theta) \\ 0 & \sin(\phi) & -\cos(\phi)\sin(\theta) \\ 1 & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

!BE AWARE!: the time derivatives of the three Euler angles $\phi, \theta, \psi$ which we write as a vector $\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$ is *not* a vector, never ever!

Likewise we can derive the angular velocities in the body fixed coordinate system as in,

$$\begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + \mathbf{R}_\psi^T \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_\psi^T \mathbf{R}_\theta^T \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} \sin(\psi)\sin(\theta) & \cos(\psi) & 0 \\ \cos(\psi)\sin(\theta) & -\sin(\psi) & 0 \\ \cos(\theta) & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \tag{9.1}$$

Why am I telling all this? In fact we are only interested in the rotational motion of a rigid body. This is governed by the set of first order differential equations of motion also unknown as the Euler equations,

$$\Sigma\mathbf{M}'_c = \mathbf{J}'_c\dot{\boldsymbol{\omega}}' + \boldsymbol{\omega}' \text{ x } (\mathbf{J}'_c\boldsymbol{\omega}'),$$

with the sum of the applied torques $\Sigma\mathbf{M}'_c$, the constant body fixed inertia tensor $\mathbf{J}'_c$, the body fixed angular velocities $\boldsymbol{\omega}'$, and the body fixed angular accelerations $\dot{\boldsymbol{\omega}}'$, all at the centre of mass $c$ and expressed in the body fixed coordinate system $x'y'z'$.

From this we can solve for the angular accelerations,

$$\dot{\boldsymbol{\omega}}' = (\mathbf{J}'_c)^{-1} (\Sigma\mathbf{M}'_c - \boldsymbol{\omega}' \text{ x } (\mathbf{J}'_c\boldsymbol{\omega}')),$$

which we can use in a numerical integration scheme to find the angular speed $\boldsymbol{\omega}'(t)$ as a function of time,

$$\boldsymbol{\omega}'(t+h) = \boldsymbol{\omega}'(t) + \int_0^h \dot{\boldsymbol{\omega}}'(t)\mathrm{d}t.$$

So much for the angular speed, but how about the orientation of the rigid body? Here we need the rotation matrix as a function of time, $\mathbf{R}(t)$. A naive approach would be to integrate $\dot{\mathbf{R}}$ but since these 9 entries have 6 dependance we are likely to get drift. Previously we found a way to parameterize the rotation matrix, where we used here the three Euler angles $\phi, \theta, \psi$, that is $\mathbf{R}(t) = \mathbf{R}(\phi(t), \theta(t), \psi(t))$. The development of the Euler angles as a function of time is found from numerical integration of their time derivatives which can be found from equation (9.1). In short, the state vector of the orientation of a rigid body is described by the Euler angles and the angular speeds. Then the dynamics is described by the time derivative of this state vector,

$$\text{State:} \begin{pmatrix} \phi \\ \theta \\ \psi \\ \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} \rightarrow \frac{\text{dState}}{\mathrm{d}t} = \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\omega}'_x \\ \dot{\omega}'_y \\ \dot{\omega}'_z \end{pmatrix}$$

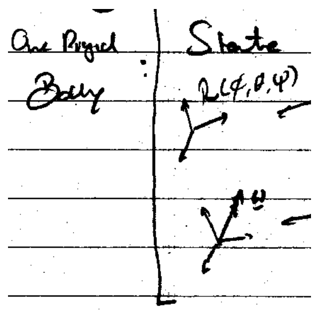


Figure 9.8: The full state for the orientation of a rigid body.

We call the time derivatives of the state, the state equations. For the Euler angles the state equations can be found from (9.1), by solving for the Euler angle rates,

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \frac{1}{\sin\theta} \begin{pmatrix} \sin(\psi) & \cos(\psi) & 0 \\ \cos(\psi)\sin(\theta) & -\sin(\psi)\sin(\theta) & 0 \\ -\sin(\psi)\cos(\theta) & -\cos(\psi)\cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix}$$

Note two things. First, the Euler angle rates depend in a nonlinear way on the Euler angles itself. Second, for $\sin(\theta) = 0 \rightarrow \theta = 0 + k\pi$ the matrix is singular and we can not solve for the Euler angle rates. Which seems odd. Let us look

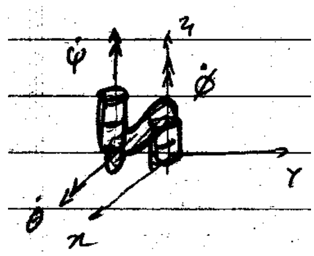

Figure 9.9: Singular $z$–$x$–$z$ Euler angle configuration, aslo known as Gimbal lock.

at such a singular configuration where $\theta = 0$ and $\psi = 0$ (the latter condition is not necessary for a singular configuration but makes things somewhat more easy to picture), equations (9.1) then become,

$$\begin{pmatrix} \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

Clearly from these equations we see that they can only be true if $\omega'_y$ is zero. Which we can easily see from the singular configuration depicted in Figure 9.9, where the middle hinge ($\theta$) aligns the two other hinges and by such confines the angular speed in the $xz$-plane. The general solution of the Euler angle rates in this singular configuration is, $\dot{\theta} = \omega'_x$, $\omega'_y = 0$ and $\dot{\phi} + \dot{\psi} = \omega'_z$. Such a singular

configuration of the Euler angles is sometimes referred to as Gimbal lock.
One way to get rid of the singularity is to redefine the Euler angles by changing $\theta = \theta - \pi/2$. A major disadvantage of this reformulation of angles is that new

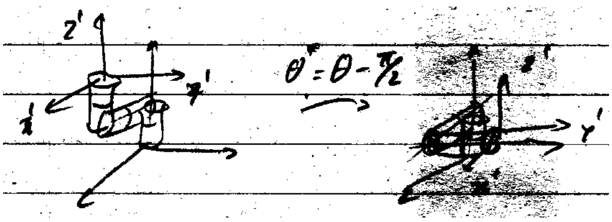


Figure 9.10: Avoid the Euler angle singular configuration (Gimbal lock) by redefining $\theta = \theta - \pi/2$.

$\phi, \theta$ and $\psi$ are not smooth in time anymore. This is very disadvantages for the process of numerical integration which builds mainly on smooth functions.

## 9.3 Euler parameters

There is a solution to all these problems. Euler's Theorem on rotation of a rigid body in space reads,

> Any rotation in 3D space can be described by a rotation about a fixed axis at a given angle.

Such a rotation is depicted in Figure 9.11, where we used for the rotation axis the unit vector $\hat{\mathbf{h}}$ and the angle of rotation is $\mu$. A point $p'$ is transformed by this rotation into point $p$. We now would like to find this transformation (the rotation matrix!) in terms of this axis–angle description. We start by looking

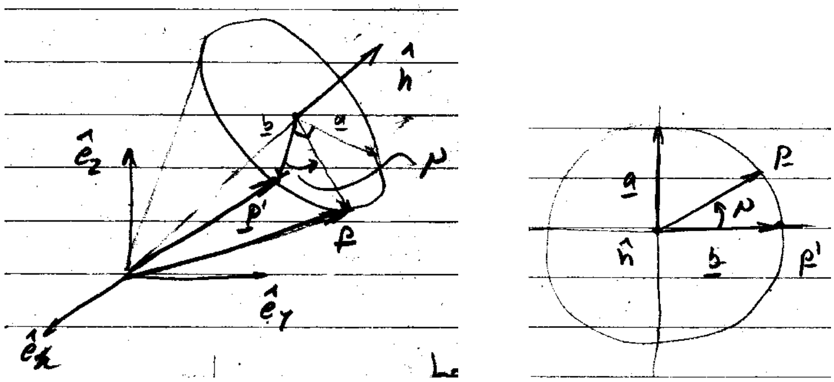


Figure 9.11: Rotation in space about the fixed axis $\hat{\mathbf{h}}$ over an angle $\mu$, where point $p'$ is transformed to point $p$ (left), and top view of this rotation (right).

on the top of the cone, Figure 9.11(left) and make an orthogonal coordinate system $(\mathbf{a},\mathbf{b})$, where

$$\begin{aligned}
\mathbf{a} &= \hat{\mathbf{h}} \times \mathbf{p}' \\
\mathbf{b} &= \mathbf{a} \times \hat{\mathbf{h}} \\
&= \left(\hat{\mathbf{h}} \times \mathbf{p}'\right) \times \hat{\mathbf{h}}
\end{aligned}$$

Now express $\mathbf{p}$ in terms of $\mathbf{p}'$ with the use of these two orthogonal vectors $\mathbf{a}$ and $\mathbf{b}$ and the angle of rotation $\mu$,

$$\begin{aligned}
\mathbf{p} &= \mathbf{p}' - (1-\cos(\mu))\,\mathbf{b} + \sin(\mu)\mathbf{a} \\
&= \mathbf{p}' - (1-\cos(\mu))\left(\hat{\mathbf{h}} \times \mathbf{p}'\right) \times \hat{\mathbf{h}} + \sin(\mu)\left(\hat{\mathbf{h}} \times \mathbf{p}'\right) \\
&= \mathbf{p}' + (1-\cos(\mu))\,\hat{\mathbf{h}} \times \left(\hat{\mathbf{h}} \times \mathbf{p}'\right) + \sin(\mu)\left(\hat{\mathbf{h}} \times \mathbf{p}'\right)
\end{aligned}$$

Now use,

$$\begin{aligned}
\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\
\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha)
\end{aligned},$$

to write,

$$\begin{aligned}
\sin(\mu) &= 2\sin(\mu/2)\cos(\mu/2) \\
1-\cos(\mu) &= 2\sin^2(\mu/2)
\end{aligned}$$

and substitute these in the previous expressions, which results in,

$$\mathbf{p} = \mathbf{p}' + 2\sin^2(\mu/2)\left(\hat{\mathbf{h}} \times \left(\hat{\mathbf{h}} \times \mathbf{p}'\right)\right) + 2\sin(\mu/2)\cos(\mu/2)\left(\hat{\mathbf{h}} \times \mathbf{p}'\right).$$

Instead of using the unit axis $\hat{\mathbf{h}}$ and angle $\mu$ we introduce a new set of parameters,

$$\lambda_0 = \cos(\mu/2) \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \sin(\mu/2)\hat{\mathbf{h}} \ ,$$

the so-called Euler parameters. These are four parameters, $\lambda_0, \lambda_1, \lambda_2$ and $\lambda_3$, where we usually write the first as a scalar $\lambda_0$, and the last three as a vector $\boldsymbol{\lambda}$. Note that theses four parameters show a dependency, namely since we assume the axis of rotation $\hat{\mathbf{h}}$ to be a unit vector, the norm of the Euler parameters should be equal to one,

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \cos^2(\mu/2) + \sin^2(\mu/2) = 1.$$

This is a constraint we have to impose on the four Euler parameters. That we have to impose a constrained is obvious since rotation is space can be defined uniquely by three parameters, ergo one constraint on the four Euler parameters. Substitution of these Euler parameters into the previous expressions gives us,

$$\mathbf{p} = \mathbf{p}' + 2\boldsymbol{\lambda} \times (\boldsymbol{\lambda} \times \mathbf{p}') + 2\lambda_0\,(\boldsymbol{\lambda} \times \mathbf{p}').$$

We can write this in matrix-vector form with the help of the tilde notation for the vector cross product $\boldsymbol{\lambda} \times \mathbf{p}' = \tilde{\boldsymbol{\lambda}}\mathbf{p}'$, where

$$\tilde{\boldsymbol{\lambda}} = \begin{pmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{pmatrix}$$

resulting in,

$$\mathbf{p} = \left(\mathbf{I} + 2\tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{\lambda}} + 2\lambda_0\tilde{\boldsymbol{\lambda}}\right)\mathbf{p}'.$$

And since by definition the rotation matrix $\mathbf{R}$ describes the transformation from body fixed to space fixed coordinate system as in,

$$\mathbf{p} = \mathbf{R}\mathbf{p}',$$

we conclude that the rotation matrix in terms of Euler parameters is given by,

$$\mathbf{R} = \left(\mathbf{I} + 2\tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{\lambda}} + 2\lambda_0\tilde{\boldsymbol{\lambda}}\right).$$

Next we expand this, where we make use of,

$$\tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{\lambda}} = \begin{pmatrix} -\lambda_2^2 - \lambda_3^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_2\lambda_1 & -\lambda_3^2 - \lambda_1^2 & \lambda_2\lambda_3 \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & -\lambda_1^2 - \lambda_2^2 \end{pmatrix},$$

which looks a lot like the inertia tensor $-\mathbf{J}_c$. Expanding all terms and making use of the constraint on the Euler parameters, $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, we get the rotation matrix expressed in terms of the Euler parameters,

$$\mathbf{R} = \begin{pmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2(\lambda_1\lambda_2 - \lambda_0\lambda_3) & 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) \\ 2(\lambda_2\lambda_1 + \lambda_0\lambda_3) & \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2(\lambda_2\lambda_3 - \lambda_0\lambda_1) \\ 2(\lambda_3\lambda_1 - \lambda_0\lambda_2) & 2(\lambda_3\lambda_2 + \lambda_0\lambda_1) & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 \end{pmatrix}$$

We know that the inverse $\mathbf{R}^{-1} = \mathbf{R}^T$ but this can also be found by counter-rotating or in other words setting $\mu = -\mu$. Then the Euler parameters become, $\lambda_0(-\mu) = \lambda_0(\mu)$ and $\boldsymbol{\lambda}(-\mu) = -\boldsymbol{\lambda}(\mu)$ and we get for the inverse

$$\begin{aligned} \mathbf{R}^{-1} &= \mathbf{I} + 2\left(-\tilde{\boldsymbol{\lambda}}\right)\left(-\tilde{\boldsymbol{\lambda}}\right) + 2\lambda_0\left(-\tilde{\boldsymbol{\lambda}}\right) \\ &= \mathbf{I} + 2\tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{\lambda}} - 2\lambda_0\tilde{\boldsymbol{\lambda}} \end{aligned}$$

This result should be identical to the transpose, as in

$$\begin{aligned} \mathbf{R}^T &= \mathbf{I}^T + 2\left(\tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{\lambda}}\right)^T + 2\lambda_0\tilde{\boldsymbol{\lambda}}^T \\ &= \mathbf{I} + 2\tilde{\boldsymbol{\lambda}}\tilde{\boldsymbol{\lambda}} - 2\lambda_0\tilde{\boldsymbol{\lambda}} \end{aligned}$$

where we used the fact that the transpose of a tilde matrix is its negative, $\tilde{\boldsymbol{\lambda}}^T = -\tilde{\boldsymbol{\lambda}}$. There are some nice anecdotes on the discovery of Euler parameters, see f.i. Altmann [1, 2].

Example: We derive the rotation matrix for a rotation about the $z$-axis over an angle $\phi$. Then, the corresponding Euler parameters are,

$$\lambda_0 = \cos\left(\tfrac{\phi}{2}\right) \quad \boldsymbol{\lambda} = \sin\left(\tfrac{\phi}{2}\right)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Substitution into the expressions for the rotation matrix results in,

$$\mathbf{R} = \begin{pmatrix} \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) & -2\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right) & 0 \\ 2\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right) & \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) & 0 \\ 0 & 0 & \cos^2\left(\frac{\phi}{2}\right) + \sin^2\left(\frac{\phi}{2}\right) \end{pmatrix}.$$

Next, using the goniometric identities,

$$\begin{array}{rl} 2\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right) & = \sin\left(\phi\right) \\ \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) & = \cos\left(\phi\right) \end{array},$$

we end up with

$$\mathbf{R} = \begin{pmatrix} \cos\left(\phi\right) & -\sin\left(\phi\right) & 0 \\ \sin\left(\phi\right) & \cos\left(\phi\right) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is clearly the rotation matrix for a rotation about the $z$-axis over an angle $\phi$.

### 9.3.1 Angular velocities and Euler parameters

The last step is to find the expressions for the angular velocity vector $\boldsymbol{\omega}$ in terms of the Euler Parameters and its time derivatives. Formally, we have $\dot{\mathbf{R}}\mathbf{R}^T = \tilde{\boldsymbol{\omega}}$. Which we expand for Euler parameters into

$$\frac{\partial \mathbf{R}}{\partial \lambda_0}\mathbf{R}^T\dot{\lambda}_0 + \frac{\partial \mathbf{R}}{\partial \lambda_1}\mathbf{R}^T\dot{\lambda}_1 + \frac{\partial \mathbf{R}}{\partial \lambda_2}\mathbf{R}^T\dot{\lambda}_2 + \frac{\partial \mathbf{R}}{\partial \lambda_3}\mathbf{R}^T\dot{\lambda}_3 = \tilde{\boldsymbol{\omega}}$$

After simplification, and with help of the identities $\lambda_i\lambda_i = 1$ (summation over $i$), we end-up with a set of three linear equations,

$$2\begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}\begin{pmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

But since the Euler parameters have this constraint $\lambda_i\lambda_i = 1$, als o the time derivatives of the Euler parameters are constrained. The corresponding constraint equation follows from the time derivative of the constraint, $2\dot{\lambda}_i\lambda_i = 0$, which we add as an extra equation to the above, resulting in a full set of four equations,

$$2\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}\begin{pmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Which we can solve for the time derivatives of the Euler parameters, $\dot{\lambda}_i$, resulting in,

$$\begin{pmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \lambda_0 & -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 \\ \lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}\begin{pmatrix} 0 \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Note that this is without any singularities, contrary to the expressions of the Euler angle rate in terms of the Euler angles and the angular velocities. Moreover, in using Euler parameters we actually never have to calculate sin's or cos's.

Finally, the expressions for the Euler parameter rates in terms of the Euler parameters and the body fixed angular velocities $\boldsymbol{\omega}'$ are found in the same manner. But we now start from the expressions $\mathbf{R}^T\dot{\mathbf{R}} = \tilde{\boldsymbol{\omega}}'$ (check this yourself). The final result is then,

$$2 \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ -\lambda_1 & \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_2 & -\lambda_3 & \lambda_0 & \lambda_1 \\ -\lambda_3 & \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix} \begin{pmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix}$$

and,

$$\begin{pmatrix} \dot{\lambda}_0 \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \lambda_0 & -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ \lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ \lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix} \begin{pmatrix} 0 \\ \omega'_x \\ \omega'_y \\ \omega'_z \end{pmatrix}.$$

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