wb1443 Homework Set #5 Answers

7.1 to do…
7.2 to do…

7.3 Experimentally.

(a)
```
[t,y] = ode23tx(inline('t^0','t','y'),[0 10],0);
err = max(abs(y-t))
err = 0
[t,y] = ode23tx(inline('t^1','t','y'),[0 10],0);
err = max(abs(y-t.^2/2))
err = 0
 [t,y] = ode23tx(inline('t^2','t','y'),[0 10],0);
err = max(abs(y-t.^3/3))
err = 1.1369e-013
% This is just roundoff error.
[t,y] = ode23tx(inline('t^3','t','y'),[0 10],0);
err = max(abs(y-t.^4/4))
err = 0.0441
% This is not just roundoff error.
```

(b) The error estimator is the difference between a second order method and a third order method. (That's why the method is called ode23.) So it is exact for f(t; y) = t^2, but not for f(t, y) = t^3. The error estimator estimates the error in the low order formula, but the function uses the high order formula to advance the solution. The function gets the exact solution, within roundoff error, for f(t, y) = t^3, but it doesn't "know" it's getting the exact solution.

Algebraically.

$f(t,y) = 1,\ y = t$

$$\begin{aligned} y_n &= t_n \\ s_1 &= 1,\ s_2 = 1,\ s_3 = 1 \\ y_{n+1} &= y_n + h(2s_1 + 3s_2 + s_3)/9 \\ &= t_n + h(2+3+4)/9 \\ &= t_{n+1} \end{aligned}$$

$f(t,y) = t,\ y = t^2/2$

$$\begin{aligned} y_n &= t_n^2/2 \\ s_1 &= t_n,\ s_2 = t_n + h/2,\ s_3 = t_n + 3h/4 \\ y_{n+1} &= y_n + h(2s_1 + 3s_2 + s_3)/9 \\ &= t_n^2/2 + h(2t_n + 3(t_n + h/2) + 4(t_n + 3h/4)/9) \\ &= t_n^2/2 + ht_n + h^2/2 \\ &= t_{n+1}^2/2 \end{aligned}$$

$f(t,y) = t^2,\ y = t^3/3$

$$\begin{aligned} y_n &= t_n^3/3 \\ s_1 &= t_n^2,\ s_2 = (t_n + h/2)^2,\ s_3 = (t_n + 3h/4)^2 \\ y_{n+1} &= y_n + h(2s_1 + 3s_2 + s_3)/9 \\ &= t_n^3/3 + h(2t_n^2 + 3(t_n + h/)^2 + 4(t_n + 3h/4)^2)/9 \\ &= t_n^3/3 + ht_n^2 + t_nh^2 + h^3/3 \\ &= t_{n+1}^3/3 \end{aligned}$$

$f(t,y) = t^3,\ y = t^4/4$

$$\begin{aligned} y_n &= t_n^4/4 \\ s_1 &= t_n^3,\ s_2 = (t_n + h/2)^3,\ s_3 = (t_n + 3h/4)^3 \\ y_{n+1} &= y_n + h(2s_1 + 3s_2 + 4s_3)/9 \\ &= t_n^4/4 + ht_n^3 + 3h^2t_n^2/2 + t_nh^3 + 11h^4/48 \\ &= (t_n + h)^4/4 - h^4/48 \\ &\neq t_{n+1}^4/4 \end{aligned}$$

7.5 (a) …

(b) Cutting the step size in half reduces the error by a factor of 2^4 = 16.

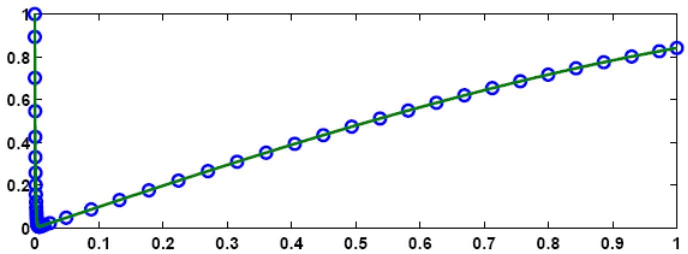
```
err1 = -2.084323879714134e-006
err2 = -1.358027112985383e-007
ratio = 15.34817574541731
```

(c) Simple harmonic oscillator.

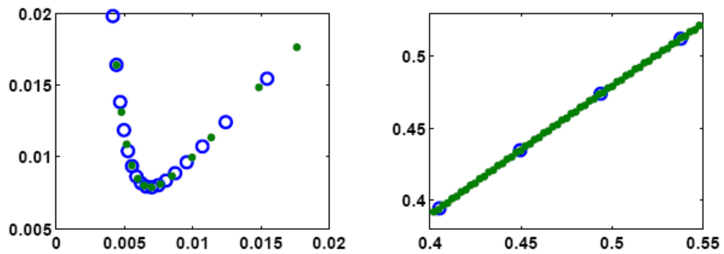
ode23	ode45	ode113	myrk4
7.20e-006	7.61e-007	1.24e-006	8.15e-007
210	30	37	100

7.6 (a) ….
(b) ode23tx requires 416 steps,
(c) while the stiff solver, ode23s requires only 57 steps.

(d)



(e,f)



7.7 (a) What is the common solution?

$$y(t) = \sin(t),\ 0 \le t \le \pi/2$$

(b) Rewrite these problems as first-order systems.

$$\dot{y} = \cos t,\ y(0) = 0$$
$$\dot{y} = \sqrt{1-y^2},\ y(0) = 0$$
$$\dot{y}_1 = y_2,\ \dot{y}_2 = -y_1, y_1(0) = 0,\ y_2(0) = 1$$
$$\dot{y}_1 = y_2,\ \dot{y}_2 = -\sin t, y_1(0) = 0,\ y_2(0) = 1$$

(c) What is the Jacobian for each problem?

$$J_1 = 0$$
$$J_2 = -2y/\sqrt{1-y^2}$$
$$J_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$J_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(d) How much work does ode45 require to solve each problem?

	f_1	f_2	f_3	f_4
steps	13	58	30	19
fevals	79	385	181	115

For the second formulation, the Jacobian $J_2 = -2y/\sqrt{1-y^2}$ becomes infinite as $t \to \pi/2$ and $y \to 1$.

(e) Change the interval to $0 \le t \le \pi$.

	f_1	f_2	f_3	f_4
steps	24	fails	60	37
fevals	145	∞	361	223

Notice that $f_2(t, y)$ is never negative, so the solution cannot decrease. At $t = \pi/2$ the theoretical solution is no longer unique. As t approaches $\pi/2$, y becomes slightly larger than 1, $\sqrt{1-y^2}$ becomes complex and `ode45` has to take impossibly small steps. The other three problems have no difficulties and require about twice as many steps as they did to reach $\pi/2$.

(f) Change the second formulation to $\dot{y} = f_2(t, y) = \sqrt{|1-y^2|},\ y(0) = 0$. For $t > \pi/2$ and $y > 1$, the equation becomes $\dot{y} = \sqrt{y^2-1},\ y(\pi/2) = 1$, The solution becomes $y = \cosh(t - \pi/2)$.

7.14. See matlab/demos/orbitode

```
te =
   0.0000
   3.0953
   6.1933
ye =
   1.2000 -0.0000 -0.0000 -1.0494
  -1.2616 -0.0012 -0.0005  1.0485
   1.1989  0.0000 -0.0047 -1.0480
ie =
   1
   2
   1
```

The events function in orbitode looks for local maxima or minima of the distance from the initial position. At t = te(1), the capsule is at its initial position and velocity, y = ye(1,:). At t = te(2), the capsule is at ye(2,:), which is its maximum distance from the initial position. At t = te(3), the capsule has nearly returned to its initial position, ye(3,:) ≈ ye(1,:). The time te(3) required to return to the initial position is the period. The initial conditions have been chosen to make this periodic orbit possible. The length of tspan is anything larger than the period.

7.15 Lotka-Volterra. ……

(a) predprey(300,150,5)

(b) predprey(15,22,6.62)

(c) predprey returns the difference between the initial and final values. Try e = predprey(102,198,alpha) for a few values of alpha between 4 and 5. alpha = 4.443 yields e = [-0.0055 0.0078].

(c) …….

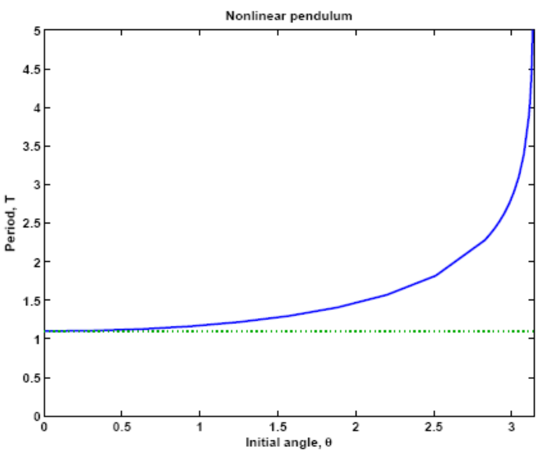
(d) Linearize around the stable equilibrium point $(r_0, f_0) = (1/\alpha, 2/\alpha)$.
These small variations are called $u = r - 1/\alpha$; $v = f - 2/\alpha$. Ignore terms of $O(uv)$.
Resulting linear system
  $du/dt = -v$
  $dv/dt = 2u$
Solutions are combinations of cos(*sqrt*(2)**t*) and sin(*sqrt*(2)**t*).
Period = *sqrt(2)***π* = 4.4429.

7.20

(a) Linearized period = 2*pi*sqrt(L/g) = 1.0988.

(b) ….

(c) The graph shows that as $\theta_0 \to 0$, $T(\theta_0) \to 1.0988$



(d)