wb1443
Matlab in Engineering Mechanics

Fall Term 2014, Thu 15:45-16:45, Mechanical Engineering, 2 ECTS credits.

Final Project 3: Eigenmotions of an Uncontrolled Bicycle.

Everybody knows from experience that a bicycle can be highly unstable at low speed, yet at moderate to high speed it can be stabilized easily. To study this behaviour we consider one of the simplest bicycle models. This model involves four rigid bodies, viz. the rear frame with the rider rigidly attached to it, the front frame being the front fork and handle bar assembly and the two knife-edge wheels, Figure 1. We assume that the wheels always stay in contact with the flat level ground and that they will not slip. Furthermore we assume a handsfree uncontrolled operation and no propulsion. Note that this makes the model energy conservative. The model has three dynamic

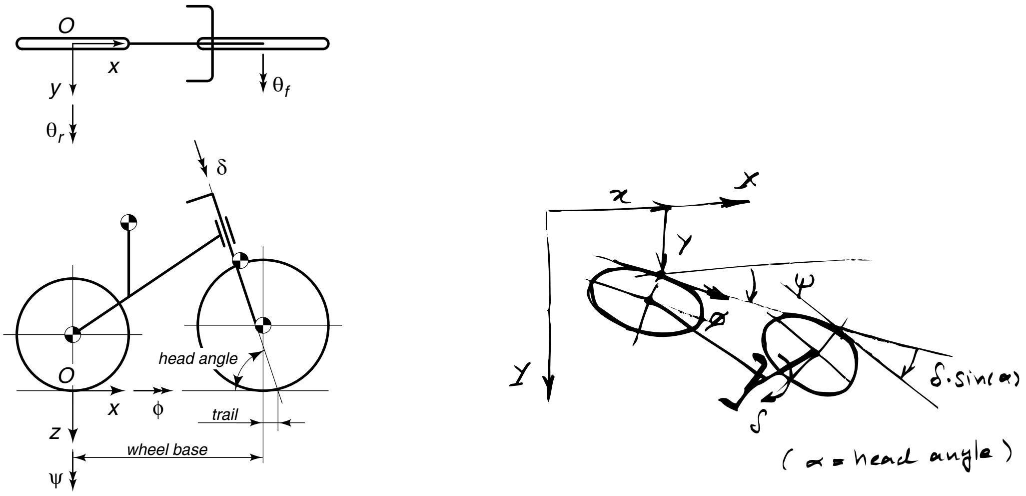

Figure 1: Bicycle model (left) together with the coordinate system, the degrees of freedom, and the parameters, and sketch (right) of a displaced position.

degrees of freedom: the roll angle ϕ of the rear frame, the steering angle δ, and the forward speed $v = -\dot{\theta}_r R_{rw}$ of the rear wheel. In our analysis we will consider only small changes to the upright steady motion of the bicycle. Then at a given forward speed v, the roll angle and the steering angle are governed by a pair of coupled second order linear differential equations of the form

$$\mathbf{M}\begin{bmatrix}\ddot{\phi}\\ \ddot{\delta}\end{bmatrix} + \mathbf{C1}\cdot v\begin{bmatrix}\dot{\phi}\\ \dot{\delta}\end{bmatrix} + \left[\mathbf{K0} + \mathbf{K2}\cdot v^2\right]\begin{bmatrix}\phi\\ \delta\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}. \tag{1}$$

The conditions of no slip at the wheels are constraints on the system which can only be expressed in terms of the velocities. These so-called non-holonomic constraints give rise to extra kinematic variables. In the case of the bicycle they are the displacements x and y of the rear wheel contact point and the yaw angle ψ of the rear frame, and are governed by the first order differential equations

$$\begin{aligned}\dot{x} &= v\cos\psi,\\ \dot{y} &= v\sin\psi,\\ \dot{\psi} &= A\dot{\delta} + Bv\delta.\end{aligned} \tag{2}$$

Parameter	Symbol	Value
Wheel base	w	1.02 m
Trail	t	0.08 m
Head angle	α	arctan(3)
Gravity	g	9.81 N/kg
Forward speed	v	*variable* m/s
Rear wheel		
Radius	R_{rw}	0.3 m
Mass	m_{rw}	2 kg
Mass moments of inertia	(A_{xx}, A_{yy}, A_{zz})	$(0.06, 0.12, 0.06)$ kgm^2
Rear frame		
Position centre of mass	(x_{rf}, y_{rf}, z_{rf})	$(0.3, 0, -0.9)$ m
Mass	m_{rf}	85 kg
Mass moments of inertia	$\begin{bmatrix} B_{xx} & 0 & B_{xz} \\ & B_{yy} & 0 \\ sym. & & B_{zz} \end{bmatrix}$	$\begin{bmatrix} 9.2 & 0 & 2.4 \\ & 11 & 0 \\ & & 2.8 \end{bmatrix}$ kgm^2
Front frame		
Position centre of mass	(x_{ff}, y_{ff}, z_{ff})	$(0.9, 0, -0.7)$ m
Mass	m_{ff}	4 kg
Mass moments of inertia	$\begin{bmatrix} C_{xx} & 0 & C_{xz} \\ & C_{yy} & 0 \\ sym. & & C_{zz} \end{bmatrix}$	$\begin{bmatrix} 0.0546 & 0 & -0.0162 \\ & 0.06 & 0 \\ & & 0.0114 \end{bmatrix}$ kgm^2
Front wheel		
Radius	R_{fw}	0.35 m
Mass	m_{fw}	3 kg
Mass moments of inertia	(D_{xx}, D_{yy}, D_{zz})	$(0.14, 0.28, 0.14)$ kgm^2

Table 1: Parameters for the benchmark bicycle from Figure 1.

Note that the kinematic coordinates do not appear in the equations of motion (1) for the lateral dynamics of the bicycle. Or, in other words, the lateral dynamics is independent of the position (x and y) and the orientation (ψ) of the bicycle on the plane.
The dimensions and mechanical properties for a typical rider-bicycle combination which were used in a benchmark are presented in Table 1. With these parameters the coefficients in the equations of motion (1) can be calculated, resulting in a mass matrix

$$\mathbf{M} = \begin{bmatrix} 80.812\,100\,000\,000\,02, & 2.323\,431\,426\,235\,49 \\ 2.323\,431\,426\,235\,49, & 0.301\,265\,709\,342\,56 \end{bmatrix}, \tag{3}$$

a "damping" matrix **C1** which depends linearly on the forward speed

$$\mathbf{C1} = \begin{bmatrix} 0, & 33.773\,869\,475\,930\,10 \\ -0.848\,234\,478\,256\,93, & 1.706\,965\,397\,923\,87 \end{bmatrix}. \tag{4}$$

a constant stiffness matrix **K0** and a stiffness **K2** which is proportional to the square of the forward speed

$$\mathbf{K0} = \begin{bmatrix} -794.119\,500\,000\,000, & -25.739\,089\,291\,258 \\ -25.739\,089\,291\,258, & -8.139\,414\,705\,882 \end{bmatrix}, \mathbf{K2} = \begin{bmatrix} 0, & 76.406\,208\,759\,656\,57 \\ 0, & 2.675\,605\,536\,332\,18 \end{bmatrix}, \tag{5}$$

and finally the constant A and B from the kinematic equations (2)

$$A = 0.074\,406\,533\,180\,43, \text{ and } B = 0.930\,081\,664\,755\,41. \tag{6}$$

Throughout the assignment we assume the SI units kg, m, sec, and rad.
To investigate the stability of the lateral motion we assume solutions of the form

$$\begin{bmatrix} \phi \\ \delta \end{bmatrix} = \begin{bmatrix} \phi_0 \\ \delta_0 \end{bmatrix} \exp{(\lambda t)}. \tag{7}$$

Substitution of these solutions in the equations of motion (1) leads to a non-standard eigenvalue problem. The solution of this eigenvalue problem yields the eigenvalues λ_i and the corresponding eigenvectors $[\phi_0; \delta_0]_i$, where i runs from 1 to 4. Some of these eigenvalues can be a complex conjugated pair giving rise to an oscillatory motion, where others are real, resulting in an exponential decaying or increasing motion. The stability of all these eigenmotions is governed by the real part of the eigenvalue: stable when negative, unstable when positive.
The non-standard eigenvalue problem is unsatisfactory, hence:

a. Rewrite the equations of motion (1) as a set of first order differential equations by introducing two new variables ρ and and ω, being the time derivatives of ϕ and δ.

b. Now use these first order differential equations to determine, for the standard bicycle, the stability of the lateral motion in a forward speed range of $0 \leq v \leq 10$ m/s. Therefore substitute in these set of equations solutions of the form

$$\begin{bmatrix} \rho \\ \omega \\ \phi \\ \delta \end{bmatrix} = \begin{bmatrix} \rho_0 \\ \omega_0 \\ \phi_0 \\ \delta_0 \end{bmatrix} \exp(\lambda t). \tag{8}$$

This will lead to a standard eigenvalue problem. Solve the eigenvalue problem for a number of points (50 or 100) in the desired speed range. Plot the real part of the eigenvalues λ as a function of the forward speed v and mark the forward speed range for which the lateral dynamics is stable. Plot in the same figure but with another line type the imaginary part of the eigenvalues λ as a function of the forward speed v.

c. Determine the lower and upper bound of this stable forward speed range with at least 3 significant digits.

Below a forward speed of 0.5 m/s we have four real eigenvalues, two positive and two negative ones. Above 1 m/s we have one complex conjugated pair and two real eigenvalues. The complex pair corresponds to the so-called weave motion of the bicycle. This initially unstable motion becomes stable with increasing speed. The moderate negative real eigenvalue belongs to the so-called capsize motion of the bicycle. The capsize is initially stable but becomes mildly unstable with increasing speed.
Next we want to visualize these motions. This is no problem for the real eigenvalues. The entries in the eigenvector determine the amplitude ratios of the degrees of freedom. Unfortunately for the complex conjugated eigenvalues this is not straight forward. The corresponding eigenvectores are in general complex. This means that besides the amplitude difference there is also a phase difference between the degrees of freedom.

d. Investigate the eigenmotions by looking at the eigenvectors and try to find a way to depict the motion by means of some sort of displaced bicycle figure. This should be done for a number of forward speeds, preferably for $v = [0, 1, 2, 4, 5, 6, 7, 10]$ m/s. Explain the naming 'weave' and 'capsize'.

So far we have only paid attention to the eigenmotion with respect to the lean angle and the steering angle of the bicycle. Finally we would like to visualize the complete eigenmotion of the bicycle which involves the inclusion of the kinematic coordinates.

e. Visualize the complete eigenmotions of the bicycle. One could think of making a movie or generating a number of 'stills' were the camera moves with a constant forward speed v. This should be done for a number of forward speeds, preferably for $v = [0, 1, 2, 4, 5, 6, 7, 10]$ m/s

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