

PROVISIONAL DRAFT

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STABILITY OF THE BICYCLE

From "Theorie des Kreisels" (Theory of Gyroscopes) by Sommerfeld and Klein, 1903 edition available from University Microfilms, Ann Arbor. [See also "Mechanics", A. Sommerfeld, Academic Press, p.156]

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[Notes in square brackets by Jim Papadopoulos, Cornell Bicycle Research Project, who edited the translation.] and typed

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Section 8: The Stability of Bicycles

The most essential question concerning the construction of modern bicycles, that of the energy which the rider must provide in order to achieve reasonably great velocities, has been treated theoretically in many ways.¹ Greater energy savings, and easier steerability, are essential advantages of the bicycle over the tricycle. However, with these advantages is associated a drawback, in that a vertical bicycle is in unstable equilibrium. To maintain this equilibrium in the presence of disturbing influences, it is necessary to learn to ride a bicycle, which is facilitated by the special construction of modern bicycles.

Is the gyroscopic action of the rotating wheels a significant factor in the stabilisation of the upright position? In the normal way of describing a gyroscope, we say that the axis of rotation tends to maintain a fixed direction in space. Considering the small mass of the wheels compared to the mass of the whole system formed by the rider and bicycle, one might doubt such an effect. And, it is obviously it is not the manufacturer's intention to increase such an effect, because his goal of saving energy leads him to build all the parts as light as possible, whereas an increase of the mass of the wheels would lead to better gyroscopic stabilisation.

In any case, we want to emphasise at this point that the gyroscopic action can be effective only if the system has sufficient degrees of freedom.

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For the unicycle, that is a single rolling disk, gyroscopic stabilisation is undoubtedly possible, as is shown by theoretical calculations in agreement with experience. [Note: for passive pitch stability with no trailing wheel the rider's c.m. should be below wheel axis.] At a sufficient velocity, the rolling motion of such a disk is stable in a vertical plane.² In this case, the motion can be viewed as an advancing motion of the center of mass, combined with a rotation about the center of mass. The latter was shown earlier in this book to be stable.³

The closest resemblance of a bicycle to a single disk occurs in the case of the high-wheeler, which consisted of a big front wheel (corresponding to the disk), and a small rear wheel which is there to support the seat of the rider and allow control of the steering. The rear wheel would reduce the system's degrees of freedom by one, thus making stabilisation impossible, if the front part of the frame had not been mounted on a pivoting steering axis which permits the plane of the front wheel to be rotated with respect to the plane of the rear wheel. If this pivot were fixed, the whole system would have only two degrees of freedom, that of tipping about the horizontal line of the bicycle track, and the motion in the forward direction connected to the rotation of the wheels. This would preclude the possibility of stabilisation by gyroscopic action. The contemporary bicycle is different from a high-wheeler only in its dimensional relations. Both wheels have the same size, and the mass of the wheels is much smaller compared to the total mass of the bicycle. This reduces the influence of gyroscopic action.

The third degree of freedom, that of rotation around the steering axis, not only allows the gyroscopic action to stabilise the bicycle, but it also allows the unconscious action of a trained cyclist to assist in keeping

¹ Enzyklopedie der Mathematischen Wissenschaften, Vol.IV, No.9, (Walker, Spiel, und Sport), p.149

² Carvallo, Journal de L'Ecole Polytechnique, Series 2, Vol.5, 1900. [Note: an ideal rolling disk wobbles forever; did Carvallo's disk stop wobbling?]

³ [Which pages, which volume?]

Noethers role described after title page in un-numbered pgs between 759 and 761.

the bicycle upright. The original theory of the bicycle, due to Rankine,⁴ considered only these contributions of rider control actions to bicycle stability. If for example the whole bicycle tilts to the right hand side, the rider must turn the front wheel towards this side, and force the bicycle to make a right hand turn. The centrifugal force generated by this turn, which acts on the system's center of mass, has a moment about the track line, which tends to restore the plane of the bicycle to the vertical. To avoid falling to the left side, the rider must similarly turn to the left. Since the existence of the steer degree-of-freedom is necessary for this active stabilisation, it is difficult to decide how much of the stabilisation is due to gyroscopic action, and how much is due to unconscious movements of the rider.

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Against the unconditional acceptance of this theory, the rider will object that he is not aware of a continuous guidance of the handlebars, that he is able to ride securely without touching the handlebars, and that he guides the handlebars more to control the direction of the front wheel than to stabilise his equilibrium. He could furthermore generate a moment by unconsciously tilting his body sideways.

It is not yet decided to what extent the stability could be achieved by small motions of the rider, but this could be decided by an experiment. In any case, it would be interesting to investigate to what degree the self-stabilisation of the bicycle with an unmoving rider is possible, and how important gyroscopic actions are in that case. The process of stabilisation is then that one such that Rankine's steer-controlling actions are partly provided by gyroscopic effects, when bicycle construction is suitable, as we will discuss later. The question about the extent to which the bicycle is stabilised without rider actions (which means that we assume the rider to be connected rigidly to the frame of the bicycle and not touching the handlebars) was treated by Whipple⁵ and Carvallo.⁶ We will in the following investigate to what extent *gyroscopic* effects are important in stability. We will of course neglect all the peripheral issues (such as alternate-side loading of the pedals, flexibility of the tires which results in a finite area of contact, friction in the steering pivot, and twisting friction of the tire on the ground).

Whipple and Carvallo set out the general Lagrange equations of the first and second kind, the latter being suitably modified because it is not a holonomic system, and they are specialised to the case of small oscillations about straight-line upright motion. [In fact, Whipple gave a purely Newtonian analysis.] We hope to be able to make the mechanical connections more clear, if in the derivation of the approximate equations, we add to the forces on the system at rest the gyroscopic action and centrifugal forces generated by the motion. This was also done in the applications we treated earlier. To obtain the terms to first order in a small oscillation, it is in this case sufficient to use the simplified expression I of the gyroscope expression on page 764. [*** Get this!]

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If we neglect terms that are quadratic in small oscillations, we remark that the size of the deviation considered are completely within the limit for which this approximation is valid.

The equations obtained in this manner agree with the equations of Whipple and Carvallo. [We agree with Carvallo's equations, but have not yet been able to confirm Whipple's.] There is one remark to make. The motion is of course unstable for small velocities, whereas at certain intermediate velocities the motion becomes stable. This means that the oscillations can be written in the form

$$Ae^{\lambda t}$$

where λ is a complex quantity with a negative real part. Whipple, under numerical assumptions for bicycle parameters which agree somewhat better with modern practice than those of Carvallo, finds stability between 16 km/hr and 20 km/hr, which are easily achievable. In what appears to be a paradox, for higher velocities the motion becomes unstable, but this can easily be explained in terms of how the parts of the system are coupled. It is useful furthermore that the lateral instability is only weak, and can be suppressed by almost unnoticeable movements of the rider even when he is not touching the handlebars.

For us the contribution of gyroscopic action to these results is interesting. We will show something that was not followed up by the cited authors: that in the absence of gyroscopic actions, the speed range of

⁴ Rankine, The Engineer, Vol.28, pp.79,129,153,175; 1869.

⁵ Whipple, Quarterly Journal of Pure and Applied Mathematics, Vol.30, No.120, p.312, 1899

⁶ Carvallo, Journal de L'Ecole Polytechnique, Series 2, Vol.6, 1901

complete stability would vanish. The gyroscopic action, in spite of its smallness, is essential for autonomous (passive) stability.

The bicycle (Fig. 135) consists essentially of a frame which carries the rear wheel in its central plane, and a steering assembly with an axle similarly carrying the front wheel. The axis is supported by a tube in the plane of the frame, so that in effect we have two plane systems hinged together at their line of intersection. The rider is rigidly attached to the frame. The steering axis in modern bicycles is tilted backwards, and its geometry is such that the extension of the steering axis cuts the vertical diameter of the front wheel at a point between the wheel center and the ground contact point.

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As Bourlet⁷ stresses, this arrangement of the axis makes it more difficult for the front wheel to turn (collapse) sideways if the rear frame is held upright. [Is Bourlet saying that negative trail tends to make the wheel flip around when in motion?] A closer investigation of the consequences of this not unimportant arrangement is not necessary here, because its influence will be made clear naturally in the course of our analytical treatment.

Because of the kinematic connections between the front and rear wheels, we can at any instant define the position of the frame by the two co-ordinates at which the rear wheel touches the ground, plus the angles of heading and lean. As well, we have a co-ordinate defining the rotation of the steering assembly relative to the frame, that is the steering angle. Because of the additional condition that the front wheel must touch the ground, these five co-ordinates are sufficient to define the position and orientation of the complete bicycle. Ignoring the cyclic co-ordinates which define the rotational orientations of the wheels with respect to the bicycle, the freedom of motion is first a tilting of the plane of the frame, and then a rotation of the steering assembly about the steering axis. As the movement of both wheels must be one of rolling on the ground, the orientation of the plane of the front wheel determines its direction of motion. This in turn determines the motion of the handlebars, and also the motion of the plane of the rear wheel which is connected to the steering axis. Now, only the motion in the forward direction is not specified. The bicycle thus has three degrees of freedom.

We learn here about the characteristics of non-holonomic systems⁸,

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which include all (?) rolling systems: the Bicycle can be transferred to each of its ∞^5 possible positions with a sequence of allowed motions, but it is not possible at every instant to move directly to all nearby configurations within an infinitesimal neighborhood via an infinitesimal motion. Expressed analytically, the equations relating the variations of the five configuration co-ordinates form a non-integrable system of differential equations.

The parameters of the bicycle are as follows: M_1 is the mass of the steering assembly, and we can assume that its center of gravity S_1 is at the center of the wheel, without much error. [Note: this is badly wrong if there is luggage or a motor on the steering assembly.] Its height h_1 is thus the radius of the front wheel, as shown in Fig.135. We are ignoring the handlebars and front fork, which would displace the center of mass from the center of the wheel, though of course we will at least add their mass to the mass of the wheel at S_1 . M_2 designates the mass of rear wheel, frame, and rider. The center of gravity S_2 is at height h_2 , and is at a distance r in front of the point B_2 where the rear wheel touches the ground.

Further, let A_v be the moment of inertia of the steering assembly about the vertical axis through the point B_1 where it contacts the ground, and let A_h be its moment of inertia about its track-line (the horizontal line where its plane intersects the ground). B_v , B_h are the corresponding quantities of the system consisting of the rear wheel+frame+rider, about the rear contact point B_2 , and B_{hv} is its product of inertia.

The steering axis has a tilt of σ with respect to the vertical, and the point where it intersects the ground lies a distance c_1 in front of the front contact point, and a distance c_2 in front of the rear contact point. So, $c_2 - c_1 = l$, where l is the length of the wheelbase.

Furthermore let θ_2 be the tilt of the plane of the rear wheel away from the vertical (positive towards the right hand of the rider), θ_1 is the corresponding tilt of the front wheel, and γ is the angle between front and rear wheel measured around the steering axis, (positive if the front wheel is turned to steer to the left side

⁷ Bourlet's interesting small book Nouveau Traité des Bicycles et Bicyclettes, Paris 1898, p.87

⁸ Hertz, Die Prinzipien der Mechanik, I. Buch, Abschn.4, Nr. 123-133

of the rider). Let ϕ_1 be the angle of the line formed by the intersection of the plane of the front wheel with the ground plane, measured in the same sense as γ ; ϕ_2 is the corresponding angle for the rear wheel. Here we assume that these angles ($\theta_1, \theta_2, \phi_1, \phi_2, \gamma$) are small, as only in this case do these definitions have an immediate meaning. Under these limitations, we have to establish the kinematic formulas for the motion.⁹

We think of the front wheel being brought to its tilted position θ_1 in two steps.

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First the whole bicycle is tilted to the angle of the rear wheel θ_2 , with no steering allowed, then the rotation γ about the steer axis is added. [Note: this causes the front of the frame to rise.] The latter we can decompose into its horizontal and vertical components (see Fig. 135) to give a tilt of $-\gamma \sin(\sigma)$ about the track-line (taking account of the above-defined senses of rotation). This can be added, according to the theorems of small rotations, to the first tilt, yielding

$$\theta_1 = \theta_2 - \gamma \sin(\sigma).$$

The vertical component of the rotation γ then lets us write

$$\phi_1 - \phi_2 = \gamma \cos(\sigma).$$

To these geometric relations between the orientation co-ordinates there is added according to the above remarks a nonholonomic relation for their time derivatives. Let the average velocity of the forwards motion be u . We assume that the orientation of the bicycle, the velocity u , and the rate of rotation $\frac{d\phi_1}{dt}$ of the front wheel about its contact point are given. The desired relations are obtained if we observe that these two conditions determine the motion of the point where the steering axis intersects the ground. The motion of this intersection is determined, as is the motion of the plane of the rear wheel, which follows the steering axis. The intersection point F (fig. 136) has along the trace of the front wheel the velocity u . Perpendicular to this line it has the velocity $c_1 \frac{d\phi_1}{dt}$. We form the component of the motion perpendicular to the track of the rear wheel which forms the angle $\phi_1 - \phi_2 = \psi$ with the track of the front wheel. So this component becomes:

$$c_1 \frac{d\phi_1}{dt} \cos(\phi_1 - \phi_2) + u \sin(\phi_1 - \phi_2),$$

or, to first order:

$$c_1 \frac{d\phi_1}{dt} + u(\phi_1 - \phi_2).$$

As a result of the rotation $\frac{d\phi_2}{dt}$ of the rear wheel the point F also has the perpendicular component $c_2 \frac{d\phi_2}{dt}$, so setting them equal we finally arrive at

$$c_2 \frac{d\phi_2}{dt} = c_1 \frac{d\phi_1}{dt} + u(\phi_1 - \phi_2)$$

as the nonholonomic relation we seek.

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Because c_1 is small compared to c_2 , the above condition expresses the rear wheel in general having to follow to the side towards which the front wheel is turned. [Note: if they are equal, steering becomes impossible — get tractrix convergence only.] In consideration of the component of the point F's velocity in the track of the rear wheel, another condition for the forward velocity of the rear wheel will follow, which shows that it differs from u by terms of second order. For our purposes, this condition is unimportant.

Now we have to determine the forces and reactions acting on the bicycle (fig. 137). At the center of gravity S_1 acts the force $-M_1 g$, while at the center of gravity S_2 acts the much larger force $-M_2 g$. Reaction forces act at both contact points and at the steering axis connecting the front and rear parts of the frames.

⁹ For an exact treatment of the kinematics, see Whipple or Carvallo loc. cit. [Note: it would be surprising to find an exact treatment because of the difficulties in calculating precisely the drop of the front of the frame due to the turning of the handlebars.]

Let's consider first the vertical reactions. We only need to consider reactions which balance the gravity forces on an upright bicycle travelling in a straight line, as these are terms of finite size. Their changes in cases of small deviations from straight riding are small terms of higher order and can be neglected.

So we are dealing only with the vertical forces which support both parts of the bicycle at rest. According to the general laws, one has to assume a reaction force at the steering axis, and a reaction moment with an axis perpendicular to the plane of the bicycle. But a force and moment can be written as a single force acting at another point (fig. 137): [Note: they are saying that the front contact force is transferred vertically to act on the steering axis of the frame.]

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First if the system were completely rigid, a force

$$Z_2 = M_2 g \frac{l-r}{l}$$

acts at the rear contact point, while

$$Z_1 = M_1 g + M_2 g \frac{r}{l}$$

acts at the front one. The component $M_2 g \frac{r}{l}$ in Z_1 is transmitted through the pressure of the rear frame onto the steering assembly. Therefore we have a balance in both parts of the system connected with the steering hinge, if we add another reaction at the steering axis vertically over the point where the front wheel touches the ground:

$$-Z = -M_2 g \frac{r}{l} \text{ steering assembly,}$$

$$Z = M_2 g \frac{r}{l} \text{ frame.}$$

These are the forces acting respectively on the steering assembly and on the rear frame. The height of the point of action is $c_1 \cot(\sigma)$, as can be seen in Fig. 137.

We now come to the horizontal reactions. First a reaction $\pm Y$ at the steering axis perpendicular to the plane of the wheel, and therefore also perpendicular to Z , which stems from a transfer of sideways motion between both parts of the system. Its size, which is also of first order, is not a static quantity but depends on the state of motion. The determination of its point of action is unimportant for our purposes.

Finally, we must introduce the reactions at the steering axis in the direction of motion, as they transmit the drive from the rear wheel to the front wheel. But these are absent in a uniform straight-line ride if there is no rolling friction and in the case of small deviations they are of second order. As their lever arm around all axes in the vertical plane containing the point of contact are of first order, their moments in all the upcoming moment equations will be of third order, and can therefore be disregarded.

Furthermore we don't have to take into account that by turning the front wheel around the steering axis its contact point is displaced sideways, as geometric intuition shows. Because this displacement is first order, its influence on the considered moments around the point of contact is of second order.

We have to add a kinematic remark concerning the position of the center of gravity.¹⁰

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It is easy to show geometrically, that in our assumptions about the position of the handlebars and the steering axis, and the center of gravity S_2 , in particular $c_1 > 0$, $r > 0$, by rotating the front wheel while the rear wheel is held vertical the height of the center of gravity S_2 , in a first approximation, is unchanged for reasons of symmetry, but in calculations to second order its height is reduced. So that the position $\gamma = 0$ means a maximum of the height of the center of gravity. This means that in addition to the gravity potential due to the simple tilt to one side of the front and rear assemblies

$$\text{Const} - \frac{g}{2} (M_1 h_1 \theta_1^2 + M_2 h_2 \theta_2^2),$$

also terms with the factor $c_1 r g M_2 \gamma^2$ and $c_1 r g M_2 \theta \gamma$ have to be added. We don't have to calculate them exactly because the force terms corresponding to these enter the final equations through the reaction Z (see eqs. 6) acting on the steering assembly.

¹⁰ See Bourlet, page 91.

As in all oscillations about equilibrium positions, the additional kinetic terms corresponding to the static deflections are not considered in our first-order equations.

Besides the forces discussed so far, i.e. those we get without forward motion, we also have to add the apparent forces, i.e. the centrifugal force and gyroscopic actions. We divide the forward movement into translation with velocity u in the direction of the mean heading, and the rotation of the wheels in their planes with rotational velocity $\frac{u}{h}$. The moment of this rotation is the same for both of the wheels, and is denoted N . At the rear center of gravity S_2 the centrifugal force is

$$M_2 u \frac{d\phi_2}{dt},$$

and at the center of gravity S_1 the centrifugal force is

$$M_1 u \frac{d\phi_1}{dt}$$

is acting perpendicularly to the wheel plane.

As a result of the forward movement and consequent wheel rotation, there is a gyroscopic force whose magnitude and sign can be found by the method used for train wagons, which is described in a previous section. First there is wheel leaning rate $\frac{d\theta}{dt}$, which causes gyroscopic moments about the vertical axis, for both the rear and front wheels:

$$-N \frac{d\theta_2}{dt},$$

for the rear wheel, and

$$-N \frac{d\theta_1}{dt}$$

for the front wheel.

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As a result a lean to the right has a moment about vertical axes which tends to change the heading towards the right (as seen from the rider).

Also must be added the gyroscopic action resulting from turning about the vertical $\frac{d\phi_2}{dt}$, and acting about the track-line:

$$N \frac{d\phi_2}{dt},$$

for the rear wheel, and

$$N \frac{d\phi_1}{dt}$$

for the front wheel.

The last two moments have the same sign as the moment of the centrifugal force, and because N is proportional to u , differ only by a constant factor. The masses of both the rider and the frame contribute to the centrifugal force, but the wheels alone produce the gyroscopic moments. So the gyroscopic moments are very unimportant compared to the moment (effect) of the centrifugal force, as in the case of the train wagons (cf. p.775).

After all these preparations, we can apply the momentum equations to the degrees of freedom described earlier. For the question of stability which interests us here, only the equations for the two degrees of freedom consisting of the bicycle's lean and the steer of the steering assembly need to be considered, because the simultaneous small perturbation of the velocity in the forward direction has only a higher order effect on these two co-ordinates.

The change of momentum about the track-line of the front and rear wheels is kinetically expressed, i.e. through mass and acceleration (A_{hv} is zero, if we neglect the tilt of the steering axis [front fork?], whose mass is small and not mentioned above.) [Note: they are assuming here that the inertia tensor about the front contact of the steering assembly has vertical and horizontal principal axes — no product of inertia.] [Translation difficulty? Are they ignoring the tilt of the steering axis? Even if $\sigma = 0$ there would still be a product of inertia because of fork rake. In fact they must be ignoring the mass of the fork and handlebars, or lumping it in with the frame.]

$$B_h \theta_2'' - B_{hv} \phi_2''$$

for the rear wheel, and

$$A_h \theta_1''$$

for the front wheel.

The sum of these terms is equal to the sum of the moments about the track-line, that is the gyroscopic moments and the moment due to the front and rear weights (with lever arms h_1 and h_2 respectively), and due to the vertical reaction Z . We can make these sums algebraically rather than vectorially, because the small angle between the wheel traces has only a second order effect. Therefore we get:

$$A_h \theta_1'' + \dots$$

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Using the force Z from eq. (6), this equation becomes:

$$(7).$$

The last term of equation (7):

$$gM_2 c_1 \cot \sigma \frac{r}{l} (\theta_1 - \theta_2)$$

, or according to kinematical equation (1)

$$-gM_2 c_1 (\cos \sigma) \gamma \frac{r}{l},$$

disappears when the front- and rear-wheel planes coincide. This is according to the earlier-found lowering of the rear center of gravity due to turning the handlebars. By the way, it always disappears if the steering axis passes through the contact point ($c_1 = 0$), because to first approximation, the front wheel can be twisted freely about the steering axis without affecting the rear wheel.

We still have to set up the momentum equations for the vertical axis, but we will for simplicity set them up for an axis parallel to the steering axis, which is tilted by an angle σ from the vertical. We choose the axes through both contact points, and set up the equations (for greater clarity) for both parts of the bicycle separately. In this case we have to take reactions from the steering axis into account.

From those we only get the moment of the force $\pm Y$ that acts perpendicular to the wheel plane, and of the vertical force $\pm Z$. The sum of their moments is for the chosen axes independent of their points of action [along the steering axis?].

$$(Y + Z\theta_2)c_2 \cos \sigma,$$

for the rear wheel, and

$$-(Y + Z\theta_1)c_1 \cos \sigma$$

for the front wheel. The reaction moment at the steering axis can be neglected because both parts are freely turnable about the steering axis. [I.e., there is no torque exerted by rider?] so that this has no component around the steering axis or the axes parallel to it which we have chosen.

For the front wheel the change of momentum in that direction, according to the notation of page 868, is calculated with the kinetic elements

$$\cos \sigma A_v \phi_1'' - \sin \sigma A_h \theta_1''$$

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The gyroscopic moment about this axis comes from the gyroscopic moments on pp. 872, 873:

$$-N(\theta_1' \cos \sigma + \phi_1' \sin \sigma).$$

Also we have to find the moment due to the centrifugal force, the weight, and the Z, Y reactions. We call the lever arm of the centrifugal force $s_1 = h_1 \sin \sigma$, and accordingly we get at the rear wheel: $s_2 = h_2 \sin \sigma + r \cos \sigma$ (cf. Fig. 137). So we get the momentum equation

$$(9).$$

In the case of $c_1 = 0$, which means that the steering axis goes through the touching point of the front wheel, this equation and eq. (7) would describe the motion completely. The reactions Z and Y would drop out, because they could not influence the motion of the steering assembly about the steering axis, which actually passes through the front contact. The sideways movement of the rear wheel would be given from the movement of the front contact. Therefore the gyroscopic moments which act at the rear wheel about the vertical axis are balanced by the reaction at the steering axis, and would not be taken into account for the whole system.

But for the real arrangement of the steering axis, $c_1 > 0$, the movement of the front wheel is influenced by the movement of the rear wheel, i.e. a sideways turning of the rear wheel is transmitted to the front wheel with no sign change. [*This may be the kinematical relation enforcing lateral displacement of the steering axis, and consequent rotation of the rear, when the steering is turned.*] So for example the gyroscopic moments of the two wheels reinforce each other.

There has to be set up another equation for the rear wheel analogous to (8). The change of angular momentum is here

$$\cos \sigma (B_v \phi_2'' - \dots$$

and it becomes

$$(9).$$

In (8) and (9) we have to eliminate the reaction Y , that is calculate the variation of angular momentum about the steering axis. We thus calculate

$$c_2 \cdot (8) + c_1 \cdot (9),$$

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i.e.

$$(10).$$

Because bicycle motion is independent of the absolute values of ϕ_1, ϕ_2 , and only their difference is significant, we want to introduce the *steer angle* γ between the planes of the front and the rear wheels from the kinematical equations (2), (3), (7), (10). If we use the short notation $\gamma \cos \sigma = \phi_1 - \phi_2 = \psi$, then we get the equations

$$c_2 \phi_2' - c_1 \phi_1' = \psi u, \quad \phi_2' - \phi_1' = -\psi',$$

and because $c_2 - c_1 = l$:

$$(11),$$

hence

$$(11a).$$

With this substitution the equations (7), (10) become:

$$(I),$$

and

$$(II),$$

respectively.

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Equation (II), which is the momentum equation for the steering axis, essentially contains the actions which effect a relative turning of the two wheels. First the second term of the third line contains the actions

described by Bourlet¹¹, that is a moment that counteracts the turning of the front wheel around the steering axis if c_1 is positive, and therefore holds the front wheel upright if the rear wheel is held upright. Since the steering axis is in front of the front contact, apparently a pressure produced by the frame acts on the front wheel, whose direction is in the plane of the frame. This pressure tries to turn the front wheel about its contact, such that the planes of the front and rear parts of the bicycle move together.

The gyroscopic and centrifugal actions in the fourth and fifth lines contain terms with $\dot{\psi}$ and ψ . The latter terms are caused by the kinematical constraints which require a curvature of the path of the wheel. This path curvature is associated with a lateral force at the wheel. Notice that the terms with θ'_1, θ'_2 in the fourth line are only caused by gyroscopic action.

In the case of tilting of the bicycle, the front wheel is turned around the steering axis to the same side by the weight, because the c.m. [of the wheel] is in front of the steering axis. The first two terms of the last line express this. The last term, which contains the factor $-\psi$, is equal to the sinking (lowering) of the center of mass, which was discussed earlier, and is caused by a turn of the front wheel, if c_1 is positive. The effect of this kinematical 'context' has to be that an initial turning is enhanced by gravity.

Equations (I), (II) contain only θ_1, θ_2, ψ . We add the kinematical equations (1) as the third linear relation between these variables:¹¹

$$(III) \quad \theta_1 - \theta_2 = -\gamma \sin \sigma = -\psi \tan \sigma$$

To investigate the stability of this system, we have to take each variable as an exponential function of time

$$(12).$$

Then we obtain a fourth order algebraic equation for λ , because this is the sum of the orders of (I) and (II) ((III) has no derivative terms). There are therefore two vibrations of the system;

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these are stable if all four roots are complex with non-positive real parts, or in the case of two real roots, those are negative. If we put (12) in (I), (II), (III), we obtain after cancelling the factor $e^{\lambda t}$ three linear homogeneous equations in the factors a, b, c . The determinant Δ of the matrix of their coefficients is to be set to zero; this matrix is written

$$\Delta = \dots$$

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If one takes into account that N is proportional to u , the calculation of this determinant would yield an equation of the following form:

$$\alpha \lambda^4 + \beta u \lambda^3 + (\gamma_1 + \gamma_2 u^2) \lambda^2 + (\delta_1 u + \delta_2 u^3) \lambda + (\epsilon_1 + \epsilon_2 u^2) = 0,$$

where $\alpha, \beta, \gamma, \epsilon$ are constants independent of u . To satisfy the condition for stability, which says that this equation is not allowed to have any roots with positive real parts, it is necessary that all coefficients be positive. On the other hand it is not certain that we will be guaranteed stability just from the positiveness of these coefficients, so it is necessary to form an additional discriminant.

We don't want to do these calculations here, but refer to results obtained by Whipple, which are based on the dimensions of a modern bicycle.

For a small u the system is unstable because of its weight. According to this, we find that γ_1, δ_1 are negative, while in contrast ϵ_1 is positive, as are α, β . The coefficients of the higher powers of u , i.e. γ_2, δ_2 , are positive, in contrast to ϵ_2 which is negative, although of small absolute value. The coefficients of λ and λ^2 become positive with increasing velocity, while the λ^0 term therefore becomes negative.

Initially the coefficient of λ becomes positive for a velocity of approximately $u_1 = 12 \text{ km/hr}$, then the coefficient of λ^2 becomes positive for a velocity of approximately $u_2 = 14 \text{ km/hr}$. Finally, the λ^0 coefficient becomes negative for $u_3 = 20 \text{ km/hr}$. Stability is only possible between u_2 and u_3 , because in this interval

¹¹ Bourlet p. 90

¹¹ Our equations (I), (II), (III) are linear combinations of those from Whipple (page 323) and Carvallo (page 100).

all coefficients are positive. A closer discussion shows that we actually only have complete stabilisation in the interval between $u_4 = 16 \text{ km/hr}$ and $u_3 = 20 \text{ km/hr}$. The calculations by Carvallo, which were made for an older model of bicycle, give qualitatively the same results, but for all these limits somewhat smaller values.
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We make these results complete by showing that full stabilisation would not be possible without gyroscopic actions. For this purpose we calculate the coefficient of λ from the determinant Δ . If we use the abbreviation Mh for the weight-moment $M_1 h_1 + M_2 h_2$, this becomes

$$g(-M_1 h_1 c_1 + \dots,$$

which reduces to

$$(13).$$

In this expression, the last term contains u^3 because N is proportional to u , while all the other terms only contain the factor u . Of these terms proportional to u , the negatives outweigh the positives, because the last term contains the small factors c_1, r , and therefore for small velocities u the whole coefficient is negative. It would remain negative, and therefore upright motion would be unstable, if the gyroscopic actions (*i.e.* spin angular momentum) were neglected — $N = 0$. (That is, since the rotational velocity of the wheels is proportional to u , we would be neglecting the wheels' polar moments of inertia). As an effect of the last term (u^3), which is caused by gyroscopic action, the coefficient becomes positive with a sufficiently high velocity. (The approximate dividing line between 'small' and 'high' velocities can be taken from Whipple's values above, *i.e.* about 12 km/hr .)

The stability of the bicycle found by Whipple for velocities between 16 — 20 km/hr is therefore only possible with the gyroscopic effects of the rotating wheels.

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The purpose of the following remarks is to explain how the gyroscopic effects really work. The factor N in the last bracket of the previous equation,

$$(14),$$

arises from the gyroscopic action $-N \frac{d\phi}{dt}$ about the vertical axis. This force, which following a tilt of the bicycle turns the front wheel to the correct side, is required for stabilisation. In the other part of the expression,

$$2Nu + Mhu^2,$$

the quantity $2Nu$ arising from gyroscopic action $N \frac{d\phi}{dt}$ which tends to hold the bicycle up only adds to the much larger moment of centrifugal force, Mhu^2 , and is therefore negligible.

The stabilising action of rotation therefore arises from the fact that the bicycle, once it is tilted sideways, is forced by the gyroscopic forces of the front wheel to enter into a curve, therefore triggering a centrifugal force which tends to right the bike. The proper stabilising force, which overwhelms the force of gravity, is the centrifugal force, and the gyroscopic action plays the role of a trigger. By the way, because of the factor $\cos\theta$, the ability to stabilise increases when the steering axis is brought nearer to the vertical.

However, there is also gravity acting on the front wheel, and the reaction force Z (of the rear of the bicycle acting on the front of the bicycle), which together act in such a way that once the bicycle is tilted to one side, the front wheel is steered to that side, which again triggers a righting centrifugal force. In spite of this, as pointed out in the preceeding, this action is not capable of totally stabilising the system. The gyroscopic effect is the only force which is proportional to $\frac{d\phi}{dt}$, whereas the gravity moment is proportional to θ itself. Therefore, after the first steering the derivative-force follows a tilting action faster, so to speak, than the gravity action. The gyroscopic action is in fact shifted by a quarter-period relative to the sinking of the frame, but the other is shifted by a half-period.

To understand the reasons for the eventual (high-velocity) instability, we now consider the λ^0 term $\epsilon_1 + \epsilon_2 u^2$ in the equation $\Delta(\lambda) = 0$. This term contains, as one can see from the matrix Δ , terms with the factor g^2 , which don't contain u . The sum of these terms, namely

$$(15),$$

is positive.

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Here the two negative terms are considerably smaller than the positive terms because of the factors $c_1 r$, $c_1^2 r^2$. In addition there are terms which contain factors of u^2 ,

$$(16).$$

Here the negative term $-gNuM_2h_2\sin\sigma$ dominates, which contains the weight of the bicyclist himself; the last term is insignificant because of the small factors c_1, r . In addition the terms of (16) will dominate the terms of (15) when u is large. Therefore the total coefficient becomes negative for large velocities, and the bicycle becomes unstable.

Searching for the origin of the terms in (16), these terms correspond to terms with the factor $u^2\psi$ in the equation II, and they arose from the introduction of the kinematic equations

$$(2), (3).$$

These equations express the virtually obvious fact that in the case where the planes of the two wheels are tilted relative to each other, aside from external forces, the plane of the rear wheel is continuously moving towards (i.e. aligning with) the plane of the front wheel, because the plane of the rear wheel has to follow a line set by the front wheel. If one imagines that the front wheel is dragged along in a [straight?] path, then the track of the rear wheel will approach asymptotically the track of the front wheel (i.e. as in a tractrix). At large velocities, this qualitative behaviour (merging of planes) occurs very rapidly, so that for $u = \infty$, from the kinematical condition (3) (excluding the case where both oscillations of the front wheel are very rapid [Do they mean eigenvalues must be small, or that the oscillation must not be excited?]), it follows that

$$\psi = \phi_1 - \phi_2 = 0.$$

Then in this case the bicycle acts as if the front and back parts are rigidly connected.

But the stabilising which comes from the gyroscopic actions is rooted in relative tilting of two wheels (i.e. in steering of the bicycle). As soon as the two wheels are coupled in a rigid manner, we can compare the whole system with a simple top which does not have 3 degrees of freedom, but because of the two contacts with the ground, has only two,

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and then we can apply a general principle which we outlined in Section 1, that as soon as you eliminate one degree of freedom, any kind of gyroscopic stabilisation is going to disappear. This explains the initially somewhat paradoxical observation, that for large velocities the bicycle's gyroscopic stabilisation fails. In a free-spinning top, large spin rates are very favourable for stabilisation. On top of this unstable movement at very large velocities are stable oscillations, which are somewhat comparable to nutation of a free-spinning top. Because of their high frequency, we can not apply the previous conclusions to these nutation movements.

Finally we want to remark, that if the steering axis is nearly vertical, then the movement can be stable even at arbitrarily large velocities, because the last coefficient of $\Delta(\lambda) = 0$ will stay positive for small values of σ . In this case the two eigen-oscillations [I think they expect two pairs of complex-conjugate roots.] are sufficiently rapid to ensure stability in spite of the apparently rigid coupling between front and rear wheel. [Weaving is rapid enough at high speed to prevent the rigidly-coupled behaviour?] However, the more the steering axis is inclined, the more the gyroscopic stabilising effects decline, following the general behaviour when the two non-cyclic degrees of freedom (leaning and steering) approximate each other. [i.e., horizontal steering axis means front wheel can only lean?].

We want to compare these results with experience. First of all we have to point out once more that the assumption of the rider being rigidly connected to the bicycle, which underlies our whole discussion, cannot practically be realised, because a bicyclist will always perform involuntary movements which are too small to be realized consciously, but which still influence the stability of the bicycle. The rider can give two kinds of assistance to the bicycle. First he can turn the front wheel in a suitable fashion, and therefore can influence the centrifugal righting moment. The necessary steering angles are very small, just like the small angles arising from the gyroscopic effects. In addition the bicyclist, by tilting his body, can generate a gravity moment, which tends to upright a falling bicycle. [I don't like this — in a non-moving non-steering

bike, the rider can try to balance, with displacements and accelerations. On a regular bike, the rider causes the frame to lean $-\theta, \theta', \theta''$ — thus invoking both gyro precession and gravity moments.] If a bicyclist rides with his hands off the handlebars, he can not affect the bicycle by the first method and has only the second at his disposal.

Experience confirms the existence of a minimum velocity below which it is impossible to ride a bicycle hands-off, presumably because when the self-stabilising actions in a bicycle disappear, the additional assistance which one can give by tilting the body sideways is no longer sufficient. On the other hand, an upper limit of speed is not observed. The probable reason appears to be that

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corresponding to the calculations of Whipple, that the instability of a bicycle for reasonably achievable velocities is very small, so that almost imperceptible tilting of the upper body is sufficient to ensure stability.

Because of the minute assistance of a trained cyclist, we cannot prove that the bicycle's self-stability is really necessary. Therefore as far as bicycle design and construction is concerned, the object of minimising energy loss (drag) reasonably has a higher priority than providing self-stability. In spite of this one cannot really deny that the gyroscopic actions contribute to maintaining balance while riding the bike, and we would like to say that they contribute in a very intelligent way. Because of their phase relations, they sense tilting very quickly, and then put into service the much stronger, but somewhat slower, centrifugal moment, to ensure stability.

JMP notes

[Notes: Their idealisations concerning the mass distribution of the steering assembly are arguable. Their ops agree with ours. Their gyro claims are incomplete — there are designs which are quite stable with no gyro effects. Don't agree with their reasoning about nutation, and about 'rigid coupling'. I feel self-stability may be more important than they credit, having ridden unpleasantly unstable bikes.]

durch die Rollbewegungen des Schiffes übertragen werden können. Doch ist deren Wirkung durch Aufhängen des ganzen Systems an sehr langen Federn ausgeschaltet worden.

Aus Allem ist zu entnehmen, daß der Kreiselkompass in der That einen wirklich brauchbaren Ersatz für den Magnetkompass bietet. Allerdings dürften die vorläufig sehr hohen Kosten der Einführung bei der Handelsmarine zunächst im Wege stehen*) und sein Verwendungsgebiet der Hauptsache nach auf die Kriegsmarine beschränken, für die nach den eingangs geschilderten Verhältnissen ein nicht-magnetischer Richtungsweiser geradezu eine Lebensfrage bildet.

§ 8. Stabilität des Fahrrads.

Die wesentlichste Frage bei der Konstruktion der modernen Zweiräder, die nach der Energie, die der Fahrer zur Erzielung der verhältnismäßig großen Geschwindigkeit aufwenden muß, ist theoretisch vielfach behandelt worden.***) Die größere Energieersparnis, sowie die leichtere Lenkbarkeit sind der wesentliche Vorteil des Zweirades vor dem Dreirade. Mit den genannten Vorzügen ist aber der Nachteil verbunden, daß in der aufrechten Stellung der Schwerpunkt sich in labilem Gleichgewicht befindet. Um die Gleichgewichtslage bei äußeren Störungen dennoch einhalten zu können, ist eine Erlernung des Radfahrens nötig, die allerdings durch besondere Konstruktionen des modernen Fahrrads erleichtert wird.

Tragen nun zur Stabilisierung der aufrechten Lage bei genügender Geschwindigkeit auch die Kreiselwirkungen der rotierenden Räder in nennenswertem Betrage bei, indem, in der herkömmlichen Bezeichnungsweise, die Rotationsachsen bestrebt sind, ihre Richtung im Raume beizubehalten? Im Hinblick auf die geringe Masse der Räder gegenüber der Masse des ganzen, von Fahrer und Rad gebildeten Systems möchte man eine solche Wirkung bezweifeln. Und offenbar liegt es auch nicht in der Absicht des Fabrikanten, sie zu verstärken, da ja sein Bestreben dahin geht, im Interesse der Energieersparnis alle Teile des Fahrrads möglichst leicht zu bauen, während eine Vergrößerung der Radmassen der Stabilisierung durch Kreiselwirkungen zu statten käme.

Jedenfalls müssen wir auch an dieser Stelle betonen, daß die Kreiselwirkungen nur dann in Kraft treten können, wenn das System die

*) Vgl. die Diskussion zu dem eingangs zitierten Vortrag von Anschütz in der Schiffbautechnischen Gesellschaft, Jahrbuch X, pag. 361.

***) Vgl. Encyclopädie der math. Wiss. Bd. IV, Nr. 9 (Walker, Spiel und Sport), pag. 149.

genügenden Freiheitsgrade besitzt. Für ein „Einrad“, d. h. eine einzelne rollende Scheibe, ist die Stabilisierung zweifellos möglich, wie die theoretische Berechnung in Übereinstimmung mit der Erfahrung bestätigen würde. Bei genügender Geschwindigkeit ist die rollende Bewegung einer solchen Scheibe bei vertikaler Ebene stabil.*) Hier kann die Bewegung als fortschreitende Bewegung des Schwerpunkts, verbunden mit einer Kreiselbewegung um den Schwerpunkt, aufgefasst werden, und die letztere ist ja aus den früheren Untersuchungen dieses Buches als stabil bereits bekannt.

Die relativ größte Ähnlichkeit mit einer einfachen Scheibe hatten die ursprünglich vielgebrauchten „Hochräder“, bei denen zu dem grossen Vorderrad, das der einzelnen Scheibe entspricht, nur ein kleines Hinterrad tritt, um den Sitz des Fahrers zu stützen und die Lenkung zu ermöglichen. Das Hinterrad würde aber die Zahl der Freiheitsgrade des Systems um eins vermindern und dadurch die Stabilisierung unmöglich machen, wenn nicht die drehbare Lenkstange die Verdrehung der Vorderradebene gegen die des Hinterrades erlaubte. Bei Feststellung der Lenkstange hätte das ganze System nur noch zwei Freiheitsgrade, das Umkippen um die horizontale Spurlinie und die mit Raddrehung verbundene Vorwärtsbewegung, und damit fiel jede Möglichkeit der Stabilisierung durch die Kreiselwirkungen fort. Das heutige Zweirad nun ist nur in den Grössenverhältnissen von dem Hochrad verschieden; die beiden Räder sind gleich gross und die Masse der Räder ist viel kleiner im Verhältnis zur Gesamtmasse. Daher wird auch der Einfluss der Kreiselwirkungen abgeschwächt.

Der dritte Freiheitsgrad, die Drehung um die Lenkstange, ermöglicht aber nicht nur die Kreiselwirkungen, sondern auch die Hilfen, die der Fahrer zur Aufrechterhaltung des Rades selbst geben kann und die der gelernte Fahrer unwillkürlich anwendet. Die ursprüngliche Theorie des Fahrrads, die von Rankine herrührt**), berücksichtigte nur diese, vom Fahrer selbst ausgeführte Stabilisierung. Neigt sich etwa das ganze Rad auf die rechte Seite, so wird der Fahrer das Vorderrad nach eben dieser Seite drehen und dadurch das Rad zwingen, nach rechts auszubiegen. Die durch die Wendung entstehende, im Schwerpunkt angreifende Centrifugalkraft hat ein Moment um die Spurlinie, das die Radebene wieder aufrichtet. Um nun ein Überfallen nach der linken Seite zu vermeiden, muss der Fahrer die Lenkstange wieder nach links drehen usw. Gerade weil auch diese künstliche Stabilisierung die Existenz des dritten Freiheitsgrades, der Drehung um die Lenk-

*) Vgl. Carvallo, Journ. de l'Ecole polyt. 2. Ser., 5. Heft, 1900.

**) Theory of bicycle, Engineer 1869.

stange, notwendig voraussetzt, ist es schwer zu entscheiden, welcher Anteil an der Stabilierung auf die Kreiselwirkungen, welcher auf die unwillkürlichen Bewegungen des Fahrers entfällt. Gegen die bedingungslose Annahme dieser Theorie wird der Fahrer einwenden, daß er sich durchaus nicht einer beständigen Führung der Lenkstange bewußt ist, daß er ja auch, ohne sie zu berühren, sicher zu fahren im Stande ist und die Lenkstange mehr, um ein Umkippen des Vorderrads zu verhindern, führt, als zur Stabilierung seiner aufrechten Lage. Er kann ja übrigens auch durch unwillkürliche seitliche Neigungen des Körpers ein Schweremoment erzeugen, das ein Umfallen verhindert.

Es bleibe dahingestellt, inwieweit die Stabilierung durch kleine Bewegungen des Fahrers selbst erreicht werden kann, darüber könnte vielleicht das Experiment entscheiden. Für alle Fälle aber wird es von Interesse sein, zu untersuchen, in welchem Grade überhaupt Eigenstabilierung des Fahrrads ohne Bewegungen des Fahrers möglich ist, und wieweit dabei Kreiselwirkungen mitspielen. Der Stabilierungsvorgang ist dann der, daß eben die von Rankine besprochenen Hilfen teilweise die Kreiselwirkungen selbstthätig übernehmen, unterstützt durch geeignete Konstruktionen des Rades, die noch zu besprechen sind. Die Frage, wie weit das Rad ohne Zuthun des Fahrers stabilisiert ist, unter der Annahme also, daß der Fahrer starr mit dem Rahmen des Rades verbunden ist und die Lenkstange nicht in der Hand hält, ist von Whipple*) und Carvallo**) behandelt. Wir werden im Folgenden zu untersuchen haben, in welchem Grade an dieser Stabilierung die Kreiselwirkungen beteiligt sind. Dabei werden wir natürlich von allen Nebenumständen (einseitiger Antrieb durch die Pedale, Nachgiebigkeit des Pneumatik und die dadurch bedingte endliche Berührungsfläche mit dem Boden, Reibung an der Lenkstange, bohrende Reibung am Boden usw.) absehen.

Bei Whipple und Carvallo sind die allgemeinen Lagrange'schen Gleichungen erster, bzw. zweiter Art aufgestellt, die letzteren entsprechend modifiziert, da es sich um ein „nicht holonomes System“ handelt, und diese für den Fall der kleinen Schwingungen um die geradlinige aufrechte Fahrt spezialisiert. Wir hoffen, den mechanischen Zusammenhang besser hervortreten zu lassen, wenn wir zur Ableitung entsprechender Näherungsgleichungen, ähnlich wie es bei den früher besprochenen Anwendungen geschehen, zu den Kräften, die im Falle der Ruhe auf das System wirken, die von der Bewegung hervorgerufenen Kreiselwirkungen und Centrifugalkräfte zufügen. Es ist

*) Quart. Journ. of Math., Nr. 120, 1899.

**) Journal de l'Ecole polytechnique, 2. Ser. 6. Heft 1901.

dabei wieder ausreichend, um die Glieder erster Ordnung in den Schwingungen zu erhalten, den vereinfachten Ausdruck (I) der Kreiselwirkung, pag. 764, zu verwenden. Wenn wir quadratische Glieder in den kleinen Schwingungen vernachlässigen, so bemerken wir noch, daß die Größe der überhaupt in Betracht kommenden Ausschläge völlig innerhalb der Grenze liegen, für die diese Näherung bei der hier zu fordernden Genauigkeit ausreicht.

Die so zu erhaltenden Gleichungen stimmen mit denen von Whipple und Carvallo überein. Aus ihnen ist zu folgern: Die Bewegung ergibt sich für kleine Geschwindigkeiten naturgemäß als labil. Für gewisse mittlere Geschwindigkeiten aber wird die Bewegung stabil, d. h. die Schwingungen können in der Form

$$Ae^{\lambda t}$$

dargestellt werden, wo λ eine komplexe Größe mit negativ reellem Teil bezeichnet. Whipple findet unter Zahlenannahmen, die einem modernen Fahrrad besser entsprechen, als die von Carvallo, für dieses Gebiet etwa die Geschwindigkeiten von

$$16 \text{ kmh}^{-1} \text{ bis } 20 \text{ kmh}^{-1}$$

also Geschwindigkeiten, die leicht erreichbar sind. Für größere Geschwindigkeiten wird die Bewegung, was paradox erscheinen könnte, wieder labil, doch wird sich aus der Art, wie die einzelnen Bestandteile des Systems gekoppelt sind, diese Erscheinung leicht erklären. Praktisch ist übrigens die letzte Labilität nur eine schwache und kann durch fast unmerkliche Bewegungen des Fahrers, auch ohne Berührung der Lenkstange, aufgehoben werden.

Uns interessiert hier der Beitrag der Kreiselwirkungen zu den erwähnten Resultaten. Wir werden zeigen, was bei den genannten Autoren nicht verfolgt ist, daß bei Fortfall der Kreiselwirkungen das Gebiet der vollständigen Stabilität verschwinden würde, daß also die Kreiselwirkungen trotz ihrer Kleinheit für die selbständige Stabilisierung unentbehrlich sind.

Das Zweirad (Fig. 135) besteht im Wesentlichen aus dem Rahmen, der das in seiner Ebene gelagerte Hinterrad trägt, und der Lenkstange, deren Axe das Vorderrad trägt. Da die Lenkstange durch einen festen Tubus der Rahmenebene geführt ist, so handelt es sich um zwei ebene Systeme, die, um eine gemeinsame Axe drehbar, verbunden sind. Mit dem Rahmen denken wir uns auch den Fahrer starr verbunden. Die Drehaxe der Lenkstange ist bei den modernen Rädern nach rückwärts geneigt, und zwar so geführt, daß ihre Verlängerung die durch den Berührungspunkt B_1 des Vorderrads gezogene Vertikale B_1S_1

