

Historical Review of Thoughts on Bicycle Self-Stability

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A bicycle can be self-stable without gyroscopic or caster effects
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Summary

This paper is the historical motivation for the Science Magazine (April 15 2011) paper [1]

“A bicycle can be self-stable without gyroscopic or caster effects”.

The claim in that title is only interesting if people thought otherwise. Our main thesis, documented here, is that people did and do think otherwise. Most attempts to explain bicycle self-stability have appealed either to front wheel gyroscopic effects, or to caster effects, or both.

After an introductory Chapter 1, Chapter 2 surveys 140 years of bicycle self-stability thoughts and explanations. The final three chapters discuss the three most-important references in detail. Chapter 3 concerns the gyroscope theories of Felix Klein and Arnold Sommerfeld (1910) [2]. Since they rely on numbers from Whipple (1899) [3], we also checked Whipple’s calculations. Chapter 5 describes and critiques the best known, by far, of the papers about trail, that by David E.H. Jones (1970,2006) [4].

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Chapter 1

Introduction

A bicycle is self-stable at forward moving speed v if, with no rider or with a rigid no-hands rider, it can automatically recover from a sideways disturbance and balance itself. Our main aim here is to address historical observations and explanations of bicycle self-stability from the entire 140 years of the bicycle's existence. For an additional perspective, various issues related to the stable control of bicycles were recently reviewed in R. Sharp [5] (2008).

A note on abbreviations:

K&S refers to the 1910 gyroscopic paper of Klein & Sommerfeld [2] (1910)

MPRS refers to the review of the bicycle equations of motion by Meijaard, Papadopoulos, Ruina & Schwab [6] (2007)

TMS refers to the 'Two-Mass-Skate' experimental bicycle treated in the companion paper.

Note ambiguity in two common terms: *stability* and *rake*.

Stability. By 'stability' some people mean self-stability (defined above), and others mean the ease with which a rider can maintain balance, either in normal (hands-on) bicycling, or in hands-off bicycling where the rider influences the steering only by torso bending. One often can't tell if a report of, say, 'enhanced stability' refers to enhanced *self*-stability, or perhaps just easier hands-on or no-hands riding. Further, there is no universally accepted meaning to the words 'more self-stable' or 'less self-stable'. A bicycle could have a bigger self-stable speed range, could have more negative eigenvalues, could be less susceptible to perturbations, *etc.*

Another problem from the older literature is that in the 1870s and 1880s, when riders sat high over a large wheel, 'stability' was specifically used to describe the resistance to 'headers' (pitching forward over the handlebars due to a road obstruction). This is unrelated to our interest here, which is lean and steer stability.

Rake. When applied to high-wheelers in the 19th century (and still to motorcycles today), rake was defined as the angular tilt back from vertical of the steer axis. However it was measured not as an angle, but a distance forward of the wheel center from a vertical line at the steering bearing. Today, in bicycle vernacular, it means the perpendicular offset of the front wheel center from the steer axis. To avoid confusion we avoid the term.

1.1 Common beliefs

As documented in the next chapters, the most explicit common beliefs regarding bicycle self-stability are:

- A. *Front wheel gyroscope.* The front wheel must have sufficient positive (i.e., forward-rolling) spin angular momentum, so that the lean rate of falling to one side causes gyroscopic precessional steering toward that side. We will call this the ‘gyroscopic’ or ‘gyro’ effect.
- B. *Trail (caster) for coupling leaning to steering.* The front wheel must have positive trail (the wheel contact point must trail behind the line of the steer axis), so that falling to one side allows — at least in the popular view — ground contact forces to cause steering to that side.
- C. *Trail (caster) for centering.* The front wheel must have positive trail so that the wheel self-aligns behind the steering axis, like a grocery-cart caster.

Not quite as often discussed as A, B, and C above, but commonly accepted, are the beliefs that the steering axis must be tilted back toward the rider ($\lambda_s > 0$, see notation in MPRS [6]), and that a bicycle with rear-wheel steering is necessarily unstable.

1.2 Present understanding

We contrast the common beliefs above with what we now believe to be true:

1. *The gyro and trail effects are indeed relevant.* For example, if just the gyro or just the trail is zeroed, or made negative, with no other design changes, an otherwise self-stable bicycle will typically lose its self-stable behavior. In this way we agree with standard thinking.
2. *Neither gyro effects nor castor are necessary for self-stability* Bicycles can be designed that are self-stable despite one or both of these quantities being zeroed or even reversed (See the companion paper and Supplementary Online Material).
3. *Many parameters affect self-stability.* Each of many parameters can, if changed enough, entirely eliminate any possibility of any self-stable speeds. Besides gyro and caster effects, these include locations of the centers’ of mass of the rear and front frames, their moment’s of inertia, and the head angle.
4. *The factors that allow self-stability are often also relevant to no-hands rider control.* For example, both trail and spin momentum are relevant to self-stability and both allow rider torso bending to exert an influence on the steering. In other words, both self-stability and no-hands rider control authority depend on coupling between bicycle leaning and bicycle steering.
5. *Self-stability may or may not correlate with a rider’s perception of stability.* Little is known about the relation between self-stability and rider perception of stability.
6. *Bicycle self-stability is qualitatively different than self-stability of tops and rolling hoops.* In their mathematical idealizations, hoops, tops and gyroscopes are all energy conserving systems. But only the bicycle is stable in the sense the perturbations decay with time. The bicycle’s exponential asymptotic stability depends on non-holonomic contact. The rolling hoop has non-holonomic contact but its fore-aft symmetry precludes asymptotic stability (as explained in [6]).

Chapter 2

Review of thoughts about bicycle self-stability

Overview of the literature

The existence of self-stability was noted many times since the dawn of the bicycle. However even today, after repeated episodes of self-stability being widely publicized, it is not widely known to the general public, nor even to everyone with a technical interest in bicycle or motorcycle handling.

By the 1890s, the three main beliefs (A, B, C from chapter 1) had been asserted. By 1900, sophisticated and credible dynamical analyses had been performed, enough to demonstrate self-stability, but never used to yield general conclusions.

In the first half of the 20th century, interest in bicycles and even motorcycles was sparse, perhaps because of the novelty of cars and airplanes. The simpler airplane equivalent of the bicycle dynamics equations were published by G. H. Bryan [7] in 1911. Interest revived somewhat in the second half of the twentieth century, possibly due to increasing motorcycle speeds and the advent of computers.

Throughout this time, most mentions of self-stability (or rider-perceived stability) are either popular or semi-technical and not well supported by either experiments or analyses. Most explanations are based largely on analogies with shopping cart casters, rolling hoops, and gyroscopes.

In contrast, the archival technical literature addressing self-stability is limited. It includes mostly numerical studies, some experiments, and almost no analytical proofs. Sophisticated numerical explorations often may include parameter sweeps from which one can conclude, say, ‘trail exceeding 35 mm is necessary for self-stability at a forward speed of 11 m/s if all other parameters are held at fixed conventional values.’ But such a numerical result obviously cannot reveal whether trail is always necessary for self-stability.

2.1 What is new in this review

Here are some new things presented in this review.

- We have thoroughly studied what Klein and Sommerfeld (K&S [2] (1910)) did and did not say about the role of gyroscopes in bicycle self-stability. This includes discovery of a calculation error which might have contributed to K&S’s misunderstanding the generality of their result.
- We provide a full list of Whipple’s generally minor typographical and numerical errors and a translation of his example bicycle parameters into MPRS notation.

- Jones [4] (1970) provided the most compelling argument for the importance of positive trail to self-stability or ridden stability. We have found the main logical error in his idea of the extent to which gravity (in other words, potential energy) helps turn the steering of a falling bicycle. We have also related his simplified statics approach to the general dynamical equations of motion presented in MPRS [6] (2007).
- Starting from the dawn of the bicycle, we found many acknowledgments that bicycles could be self-stable. These are contrasted with some contemporary semi-technical assertions that bicycles require continuous rider input to stay upright.
- We uncovered a number of early citations claiming the importance of trail or gyroscopic phenomena to self-stability, long before Jones or even K&S.
- We also found a few isolated observations that neither trail nor steer axis tilt was as important as as been commonly asserted.
- We uncovered some early bicycle commentary by mechanics such as Kelvin, Pocklington, and Moigno. Disappointingly these have only vague statements about dynamic stability, or the ‘stiffening’ effect of spin momentum, without supporting analysis.
- We have uncovered and made available several little-known references:
 - The analytical bicycle dynamics thesis of noted dynamicist J. L. Synge (1920) [8].
 - Many of the Schwinn and government sponsored bicycle dynamics investigations performed by CALSPAN (formerly Cornell Aeronautical Laboratories) in the 1970s (weblink).
 - The bicycle control thesis of A. J. R. Doyle (1988) [9].

2.2 Some mentions of bicycle self-stability in the early literature

Before the bicycle. It’s clear that Drais, inventor of the pedal-less running machine (*Drais Laufmaschine* or *Draisine*), was fully cognizant of the ability to keep one’s feet off the ground, and coast downhill while balancing by steering [10, 11] (1819, 1832). However, no-hands riding was almost certainly impossible with a Draisine, in part because of their high-friction steering bearings. And, with no footrests they could not provide much possibility of control of hands-free balance by rider body bending. Rapidly-appearing imitators called ‘pedestrian hobby horses’ did sometimes have footrests mounted near the axle of the front wheel – see figures 28, 33, 38 in Street [12]. By this means perhaps no hands stability was possible, but only by using the feet to steer. So self-stability of pre-bicycle two-wheelers was not observed nor was it likely possible.

The earliest bicycles. The mid-1860s were the true dawn of bicycles. Pedals, low-friction steering bearings, and frame-mounted footrests were then added to the Draisine concept. Judging from illustrations and a few museum examples, such innovations as steer-axis tilt or front-fork offset (rearward, when the steer axis is vertical) had occasionally appeared on earlier non-pedaled hobby horses. But drawings in U.S. patents suggest a rapid adoption of steer axis tilt after 1865. We do not know if these design features were related to a conscious attempt to achieve self stability, or just a trend of bringing the handlebars closer to the rider.

By 1869, ‘automatic’ stability (perhaps self-stability) of bicycles had been recognized. The self-stability of a rolling hoop and of tops had been known since antiquity, and both had been

shown to follow from Newton's laws and mechanics in prior decades. Bicycles were seen as similarly self-stable and were repeatedly compared to those devices. Mostly ignored was that bicycles depend essentially on the coupling of leaning to steering. Also not observed in this era is that the self-stability of a bicycle is of a stronger type than that in the mathematical models of hoops and gyroscopes. In their mathematical idealizations, hoops and gyroscopes only have neutral stability (imaginary eigenvalues) whereas the bicycle has asymptotic (exponential) stability (eigenvalues with negative real parts).

19th Century observations in relation to stability and self-stability

The following examples are representative.

- Bottomley [13] (1868), one of several similar early bicycle chroniclers, considers bicycle balance as similar to that of a rolling hoop. He mentions no-hands riding, although his front-pedaled bicycle was then probably foot-steered. So, despite the hoop analogy, this may not be a direct observation self-stability.
- World of Wonders [14] (1869) makes an explicit identification between the hoop, the spinning top, and the velocipede. Again, despite the analogy, we can't be sure that bicycle self-stability was observed.
- Van Nostrand's Eclectic Engineering Magazine [15] (1869) states "the principle is the same ... [as a] hoop ... a velocipede might run by itself ... will take care of itself when in motion." It seems like the author suspects self-stability, but may not have observed it.
- Philp [16] (1870) writes, in a cookbook with digressions about life, writes "Riders have learnt ... on going down hilly roads, to rest with perfect safety at full length above the wheels." This presumably refers to lying back flat on the backbone of the frame. This at least seems to be a direct reference to self-stability.
- Moigno [17] (1873) states: "The bicycle is certainly an interesting application of dynamics in which a body, very unstable at rest, acquires in motion perfect stable equilibrium." This is a strong statement, but we don't know if it was backed by observation. Although he was a competent mathematician and mechanic (a collaborator with Cauchy) he provides no analysis to justify his bicycle claims.
- Nahum Salamon [18] (1876) connects bicycle self-stability to the mechanics of rotating bodies. Salamon was chairman of The Coventry Machinists Company, the leading British sewing machine manufacturer which then became a major bicycle manufacturer. It is not clear that Salamon had a sharp conception of self-stability. "It has probably never entered the minds of those who have looked upon the bicycle only as an object to invite their satire, that its motion embodies the most remarkable of mechanical facts, the permanent direction of an axis of rotation. It is that permanence that is the cause of the paradoxical stability of the motion. The same cause is in operation in the motion of a hoop, or a quoit, in all rifled projectiles, in the remarkable phenomena of the gyroscope [sic], and it is conspicuously apparent in the stupendous mechanics of the universe. Leonard Euler, the greatest mathematician of the last century, wrote a celebrated mathematical treatise, *De motu turbinis*, in which he showed that the motion of a top was applicable to the elucidation of some of the greatest phenomena of the solar system: and an instrument in which the same mechanical principle is happily made conducive to purposes of practical utility, is assuredly entitled to the merit readily accorded to other inventions similarly distinguished." Salamon was evidently embellishing, that is nearly copying but adding the word 'bicycle', to some ideas of

- the former Astronomer Royal, G.B. Airy, from whose earlier published lectures on astronomy some paragraphs on gyroscopic behavior had been summarized in John Timbs [19]. Later pages in Salamon contain an incorrect analysis of a rolling hoop, purporting to explain its stability.
- Charles Spencer [20] (1876) provides an illustration of a no-hands rider with his feet clearly on frame-mounted footrests (i.e., clear of the front-wheel pedals). “This should only be attempted by an expert rider, who can get up a speed of twelve to fourteen miles per hour, and on a very good surface, and with a good run; and, in fact, from this position you may lean back and lay flat down, your body resting on and along the spring. I know several gentlemen who could run the length of the Agricultural Hall in this way, when that spacious building was used for velocipede-riding.” Spencer was a gymnastics instructor who began selling early bicycles and evangelizing their use, see Ritchie [21]. Assuming that his riding posture does not allow much body motion, this seems to indicate self-stability.
 - Wurtz [22] (1879), a pioneering proponent of the atomic theory of matter, appealed to dynamic stability of the bicycle to justify equilibrium in the kinetic theory. It is ambiguous whether he knew of true self-stability.
 - Kelvin [23] in an 1881 lecture about a vortex-based atomic theory also used the bicycle as an example of gyroscopically-induced stability in motion. Regrettably, his statement was merely an allusion to gyroscopic stiffening, not about how gyroscopic precession might cause the bicycle to steer itself towards the side of a fall. We don’t know if Kelvin knew of true self-stability.
 - The *Wheelman* magazine [24] (1883) describes a typical riderless-bicycle stunt: “alighted and picked up a pocket-handkerchief from the floor while his bicycle went on, caught it in a few paces, and remounted.” This *is* a clear reference to self-stability, or at least to only slow instability.
 - Well-known physicist Charles Vernon Boys [25] (1884) asserted “The extraordinary stability of the bicycle at high speed depends largely on the gyroscopic action of the wheels.” Again, self-stability is possibly (but not clearly) indicated.
 - Cyp. Chateau [26] (1892) wrote “It is clear that bikes you can buy now will go on their own; and it’s totally sufficient to cross your arms and even your legs and just lean slightly to the side you wish to turn toward, and the docile instrument will follow the wishes of the rider.” This reference to easy no-hands stability with restricted upper-body motion, seems to indicate self-stability.
 - Charles E. Duryea [27] (1895), the first U.S. automobile manufacturer, observed “To test the steering of a cycle, I have stood at one side of a yard, with a friend at the other side, and we have bowled the cycle back and forth between us. If this is not mechanical [automatic steering, what is?]” This is clear self-stability, or at least only slow instability.
 - Paloque [28] (1895) specifically describes automatic steering based on various geometry, mass distribution, and gyroscopic effects. Although he mentions an automatic steering contribution, true self-stability is not stated.
 - In 1898 the MacLeods[29] observed that riding no hands with closed eyes, a state in which bicycle self-stability might play a big part, is not only possible but “beats salmon fishing altogether.”

- A tongue-in-cheek anonymous review [30] of Whipple's paper [3] in 1899 said: "The mathematician has taken to the bicycle, not so much for exercise as for a subject of infinite calculations. Among other things he has now shown that to ride easily hands off without regard to wind, stones, and car tracks it is only necessary to go at the rate of 10.4 miles an hour and to have fully inflated tires. That, at least, is what Mr. Whipple, of Trinity College, Cambridge, announces after an elaborate series of calculations to determine the relation of velocities to stability of motion."

In summary, many authors made claims of various degrees of self-stability and offered explanations for it. By 1900 there were widely publicized analytical proofs of self-stability by Whipple [3] and Carvallo [31]. Both are cited in the 1910 Encyclopedia Britannica article by Sir George Greenhill on *Gyroscopes and Gyrostats*, where bicycles are mentioned several times. (However, his bicycle balancing explanation, both in that article and in his books published around that time, recognized only the gyroscopic roll torque created when the rider actively turns the steering.) Note that this bicycle dynamics information is not presented in the Britannica entry *Bicycle*. The overall interest in bicycle dynamical self-stability is represented by this earlier commentary by Greenhill in *Nature* [32] (1899), "The next century will have its work cut out for the mathematical treatment of [the unsymmetrical top and] also the dynamics of the bicycle."

20th century self-stability citations Questions of bicycle balance were subsequently kept in vogue by two decades of discussion related to airplane handling, with explicit reference to bicycle-airplane analogies (which had been brought out as early as 1878, by Brearey [33]). Regrettably, many authors adopted a premise of bicycle instability, having seemingly forgotten the contrary evidence and calculations.

- Noted physicists Klein & Sommerfeld [2] (1910) (K&S) presented a detailed dynamic analysis similar in spirit to Carvallo, but went further to investigate the conditions for and causes of self-stability. Chapter 3 here is entirely about K&S.
- Physicist Andrew Gray built a series of gyroscopic mechanisms, including self-stable model bicycles, which appeared in various accounts [34] (1915) and in his monograph [35] (1918).
- The Westinghouse "Phantocycle" display of a riderless bicycle on rollers at the 1939 World's Fair would have been seen by millions of visitors, and was highlighted in magazines such as *Popular Mechanics* [36]. It was developed by Clinton R. Hanna, an expert in application of gyroscopes to control problems who went on to become Associate Director of Westinghouse Research Laboratory. We do not know how it functioned mechanically. On one hand a servo-system was said to adjust a biasing mass on the steering in response to deviations; and the system was said to be able to react a side-force of 3 lb, presumably applied at the seat. On the other hand the wheels ran in narrow tracks seemingly offering no opportunity for steering. In a documentary film including the bicycle [37], the front tracked roller appears to displace laterally as the handlebars steer, so perhaps there was a servo mechanism causing lateral motions of the ground contact.
- Jacques Tati's 1949 movie "Jour de fête" shows a riderless bicycle going long distances.
- An unpublished paper by Herfkens [38] (1949) discusses self-stability.
- Wilson-Jones' [39] (1951) explanation of motorcycle phenomena begins "A bicycle or motorcycle is automatically stable ..."
- Döhning [40] (1953) extended the Klein & Sommerfeld analysis to a more general front-assembly mass distribution, and performed confirming experiments.

- Occasional news accounts mention motorcycles that escaped their riders, traveling some distance and sometimes causing damage far away.

2.3 Self-stability sometimes ignored in the modern literature

After about 1965 there has been a fairly consistent stream of technical analyses calculating two-wheel self-stability. However *general* awareness of bicycle or motorcycle self-stability has seemed to fade away after each convincing proof. Despite the above-mentioned evidence and analyses, some modern commentators have seemed unaware of, or at least unconcerned with, bicycle self-stability.

- Ian Stewart [41] (1989), page 53, says “An unstable motion can be observed, but only as a transient phenomenon while the system is en route from its original unstable state to where it will finally end up. The motion of a bicycle between the time you give it a push and the moment it falls into the ditch in a final, tangled, stable rest state.”
- Jerrold Marsden [42] (1997) writes on page 14: “The problem of stabilization has also received much attention. Here the goal is to take a dynamic motion that might be unstable if left to itself but that can be made stable through intervention [aircraft comment] . . . The situation is not really much different from what people do everyday when they ride a bicycle.”
- Karnopp [43] (2004) says “These [single track] vehicles are unstable in the absence of active control of the steering either by a human operator or an automatic control system.” (p148), “Bicycles are examples of useful vehicles that absolutely require active stabilization in order to function at all.” (p228)
- Hansen [44] (2004), page 34: “a bicycle is anything but an inherently stable machine. The Wrights believed that the airplane would need to be the same sort of dynamically interactive device as a bicycle: unstable on its own, but completely controllable and virtually automatic in the hands of an experienced operator.”
- Romagnoli [45] (2006), page 10. “A bicycle is an inherently unstable system.” And “Riding a bicycle is an attempt to stabilize an unstable system . . . ”

In fairness, there are ambiguities here. For example, the above statements would be accurate for bicycles traveling too slowly to be fully self-stable. Also, as pointed out in the supplement to MPRS [6], some bicycle models used in studies of control theory are too simple to exhibit self-stability, so a neglect of self-stability is appropriate for those studies.

2.4 Early explanations for how design parameters affect self-stability or easy control

By the final 20 years of the 19th century, intelligent speculation had identified today’s three most commonly advanced mechanisms for explaining self-stability: A. gyroscopic precession for automatic steering in the direction of a lean rate; B. trail for automatic steering in the direction of a lean; and C. trail for steering alignment. In this section we summarize the literature concerning these and other potential factors in automatic steering.

But first we note that, for a manufacturer, there is a generally a qualitative difference between trail and spin angular momentum. Trail can be adjusted relatively freely by slightly altering bicycle

dimensions. But wheel moment of inertia divided by radius (the spin angular momentum per unit speed) is less freely selectable. Diameter is standardized (at about 26 inches \approx 0.7 m) so wheels can be assembled of standard parts and fit typical frames. Also, wheel mass is not a free parameter when one is trying to minimize overall weight. Thus, in distinction to trail, commentary on gyroscopic effects is aimed less at making design decisions and more at qualitative explanation, typically drawing the analogy between a bicycle and a hoop.

There have been occasional exceptions in which the gyroscopic terms are manipulated for design. For example, Kasten was awarded U.S. Patent 6918467 [46] for gearing his motorcycle front-wheel brake rotors to the wheel to make them rotate backwards at a higher speed. His intention was to reduce steer torques in rapid roll maneuvers at high speed; this modification would also increase the capsize speed. Murnen *et al.* were awarded U.S. Patents 7314225 and 7597337 [47] for a forward-rotating disk inside the front wheel of a child's bicycle, to lower the weave stability speed, and increase the steer torque needed to disturb the bicycle's lean angle.

Gyroscopic effects

It is almost universally accepted that gyroscopic effects due to the spin angular momentum of the front wheel are important for bicycle self-stability. Following are some writings where either self-stability or easy balancing is attributed to gyroscopic effects, or where bicycle stability is compared to that of a hoop or gyroscope.

- Bottomley [13] (1868)
- American Phrenological Journal [48] (1869),
- Goddard [49] (1869),
- World of Wonders [14] (1869),
- Chambers' Journal [50] (1869),
- Van Nostrand's Eclectic Engineering Magazine [15] (1869),
- Spencer [20] (1876),
- Wood [51] (1877),
- Brentano's Aquatic Monthly [52] (1879),
- Kelvin [23] (1881 lecture reported in 1889),
- English Mechanic and World of Science [53, 54] (1882).
- C. V. Boys [25] (1884).
- Boys's comments on a presentation by Griffith [55] (1886) includes the first clear statement we know of, that front-wheel spin angular momentum causes automatic steering towards the side of a fall.
- R. P. Scott [56] (1889),
- Worthington [57] (1892),
- English Mechanic and World of Science [58] (1893),

- Cyp. Chateau [26] (1892), and Paloque [28] (1895), who also pointed out that gyroscopic coupling could cause the appropriate steering in a fall.
- Archibald Sharp [59] (1896) notes the gyroscopic coupling of lean to steer, but only in the context of intentional no-hands control of steering.
- Greenhill in Nature [60] (1898) writes on the need for hyperelliptic functions to solve bicycle motion.
- MacLeod & MacLeod [29] (1898),
- Hodgins [61] (1900),
- A.G. Webster [62] (1904),
- American Machinist [63] (1913),
- Gray [34, 35] (1915, 1918),
- Grammel [64, 65] (1920, 1950),
- Wilson-Jones [39] (1951).
- More recently, Jones [4] (1970) demonstrated gyroscopic influence on a riderless bicycle, while arguing that it is of little significance in the presence of a heavy rider.

A significant demurrer is offered by Warring [66], who critiques mindless appeals to gyroscopic stiffening. He also argues that rider torso motions can play little role in balance, and then very clearly presents the idea of ‘broomstick balancing’ by translation of the support point.

Positive trail

A comparable treatment of trail is less simple. To start with, unlike gyroscopic behavior, the basic mechanics of casters was never a popular concept for the general public. Also, trail is credited with two somewhat independent effects, one coupling leaning to steering, and the other causing self-alignment of the steering. The idea that trail in a bicycle has a wheel-aligning role similar to that in a non-leaning cart – the charming German name is *Teewageneffekt* – is evident at least from the dawn of the rear-wheel-drive safety bicycle. Following are some comments about the importance of trail for bicycle self-stability.

- *Amateur Work* [67] (1881) has a long discussion of ‘castor’ in the instructions for constructing a safety bicycle.
- *English Mechanic and World of Science* [68] (1888), page 166, goes into castering plus the tendency to turn when leaned, due to the possibility of reducing potential energy. Page 345 on safety bicycles explains that forks are bent forward to reduce (but not eliminate) the trail, thus reducing the destabilizing torque due to steer while still preserving castering.
- R.P. Scott, in *Cycling Art, Energy and Locomotion* [56] (1889) includes a section on castering, including an illustration of trail and a discussion on the gravitational potential energy change in steering.
- In an article in *La Nature* (1892), Chateau [26] emphasizes the tendency of the safety bicycle (*bicyclette*) at rest to steer toward an imposed lean, due to the sloped steering axis. This factor was considered new and not applicable to the older high-wheeler (in French, *bicycle*).

- Paloque [28] (1895) was a French Army officer who wrote a multi-part article on the bicycle for military applications. Paloque has five pages on automatic steering. He specifically identifies the trail (in French *chasse*) and recommends 8–15 cm. He mentions castering for a centering tendency. He also describes, consistent with our companion Science paper, the supposed gravitational effect of forward offset of the front-assembly mass center (due to curvature of the fork) in helping steer toward the leaning side.
- A. Sharp [59] (1896), in a steering chapter, identifies both trail and forward projection of the front-assembly center of mass as part of self-stability, in the sense of recovering from a steady turn. Note, Sharp is incorrect in his assumption that static torque applied by a rider to a stable bike will have the sense of restraining the steering from straightening. (The opposite is demonstrated in the Supporting Online Material of Kooijman *et al.* [1])
- C. W. Brown in the CTC Gazette [69] (1897) describes caster action and potential energy decrease as components of easy hands-off riding.
- C. V. Boys [70], (1897), reviewing A. Sharp [59] in Nature, mentions that trail makes hands-free riding easier.
- Bourlet [71] (1899) was prolific and we have not studied all of his works. In this design-oriented book he states that the trail (and the front assembly center of mass position) makes the steering turn to the leaning side. He also says, as mentioned by K&S, that the trail makes the wheel align with the frame when it is upright.
- Garratt [72] (1899) gave a long discussion of potential energy, arguing that a design geometry that leads to no net rise nor fall at 90 degrees of steer should minimize gravitational steer torques and encourage automatic steering. He also notes that trail stabilizes the wheel's alignment with the frame. However, Garratt's implied torque calculation is defective, not only in neglecting the front-assembly weight, but also in ignoring the necessity for lean to achieve balance whenever the steering is turned (because, due to trail, steering shifts the front contact to one side of the frame). Periodically other authors (e.g., More [73] and Davison [74]) have proposed some similar condition for the same purpose.
- Whipple [3] (1899) presents calculations that show self-stability. But he did not make any comments about how a bicycle should be designed to achieve or increase self-stability.
- Hasluck [75] (1900) doesn't explain why the steer axis should be tilted and the forks bent, but his picture shows the steer axis passing above the ground contact and below the front wheel center, as was stated explicitly by Carvallo [31] in 1899.
- Bower [76] (1915) concluded analytically (but incorrectly, see Hand [77]) that trail was essential for self-stability.
- Pearsall [78] (1922) also used analysis to conclude (again incorrectly [77]) that exceeding a minimum trail was necessary for self-stability.
- Rice [79] (1974) presents five geometry criteria for selection of trail and steer axis tilt.
- Rice [80] (1976), pages 42-43: "Small values of positive mechanical trail and all negative values of mechanical trail are to be avoided in bicycle design if satisfactory free control stability is to be achieved."

- Lowell & McKell [81] (1982) explored the influences of trail and spin momentum on the self-stability of a simple bicycle (vertical steer axis, massless fork, point-mass rider). They conclude that trail is important and spin momentum is helpful. Unfortunately their ad-hoc and approximate derivation of the equations of motion did not maintain an energy-conservation symmetry of the stiffness matrix. And the out-of-plane ground reactions arising from falling were not included. (The flaws are most obvious when examining the zero speed case.) So the numerical results on which their conclusions were based are not persuasive.

We found just one clear statement suggesting that the importance of trail had been exaggerated: Wilson-Jones [39] (1951) authored a comprehensive qualitative description of factors related to self-stability. He mentions the offset mass of the front assembly, and the ground reaction acting through the moment arm of the trail (this is supposed to turn the steering in a fall, and straighten it when upright). The steering axis tilt results in energy changes due to steering when the frame is upright. He was able to ride no-hands with slightly negative trail, and concluded “trail is not so critical as is sometimes supposed”.

Steer axis tilt

Trail and steer-axis tilt are linked. If the shape (the bend) of the fork is fixed, increasing the steer axis tilt also increases the trail. Discussions prior to 1900 specifically stated that with the steer axis tilted so as to bring the handlebars close to the rider, the trail would be too large unless it were reduced by bending the fork forward, thus placing the wheel center ahead of the steering axis. Here we are concerned with the proper steer axis tilt for a fixed trail, not for a fixed fork shape.

Before about 1950, our sense is that tilt of the steer axis was only occasionally linked to self-stability. But the conventional assertion that more tilt toward the rider enhances self-stability was certainly the popular wisdom after 1960. We have not found the trigger for this thought.

- Cyp. Chateau [26] (1892) mentions axis tilt to assist automatic steering.
- The noted dynamicist J. L. Synge [8] (1921) found that a simple bicycle with mass only in its wheels could never be stable with a vertical steer axis.
- Kane [82] (1961) and later Neřmark & Fufaev [83] (1972) showed that certain simple vertical steer-axis bicycles could not be stable without a viscous steering damper.
- Tony Doyle [9, 84] (1987) used a vertical steer axis in his ‘destabilized’ bicycle designed to eliminate all coupling of roll to steer, and thereby eliminated both self-stability and the effect of torso bending on no-hands steering.
- Cocco [85] (2004) wrote “The steeper [i.e., the greater] the inclination of the rake angle, the more the motorcycle is directionally stable.”
- Stermer [86] (2007) writes “Generally, the steeper the rake the quicker the bike will steer. A less-steep rake gives the bike greater stability at speed.”

In summary, positive tilt is generally thought to increase self-stability. We uncovered only one author partially disputing this: Foale [87] (2006) describes riding experiments with a vertical steer axis, in which he found that it was less difficult than expected.

We have uncovered no discussion of a negatively tilted steer axis. Our sense is that it is tacitly assumed to be unstable, which is why we included a self-stable example with negative steer axis tilt in the Supporting Online Material of Kooijman *et al.* [1].

Position of front-assembly mass center

We include this factor, which is only rarely mentioned, because, aside from trail and gyro effects, and the tilt of the steering axis, it can be one of the most significant factors. In particular it plays a significant role in the self-stability of the TMS bicycle of the companion paper.

A frequently occurring term in the equations of motion is the front assembly mass times the perpendicular distance of the assembly center of mass from the steering axis (the first moment of mass about the steer axis). Even if the mass of the front assembly resided primarily in the wheel, it would normally have a forwards C.M. offset due to the wheel center being offset from the steer axis (due to the fork bend). Furthermore, the front assembly of a conventional bicycle usually includes forward-projecting handlebars, and can possibly have a forwards basket. And a motorcycle may have a heavy headlamp, instruments, windshield and mudguard.

- Cyp. Chateau [26] (1892) merely mentioned the effect of gravity turning the front wheel of a bicycle held at an angle
- Paloque [28] (1895) specifically defined forward projection of the front assembly mass center, in addition to trail.
- Döhring [40] added non-wheel mass to the simplistic wheel-only front assembly of the model in Carvallo and K&S. Note, the simpler K&S model did have front mass offset, but since all front mass was in the wheel, offset was simply due to the bend of the fork, which is dictated by the choice of trail. Front-mass offset was not an independent parameter for K&S.
- Neïmark & Fufaev [83] (1972) specifically examined the effect of front mass offset in their self-stability linearization. Dikarev, Dikareva and Fufaev [88] (1981) corrected a potential energy error found in [83], but then instead of repeating the original algebraic linearized study, performed a two-parameter numerical sweep for one specific base design.
- Collins [89] stated that front-mass offset was a significant parameter for his self-stability studies, that he felt had not received adequate attention in the literature.
- In contrast, Cossalter's [90] eigenvalue sensitivity studies indicated negligible effect of front-mass offset. We surmise that this is an artifact of Cossalter's 'ten percent delta' approach, on a motorcycle with small initial offset.

Rear steering and self-stability

Rear-wheel steering is conventionally considered to make a bicycle unrideable. But in our investigations for the companion paper we were able to exhibit a self-stable rear-steered design. So we explored the literature for statements about it.

Rear steering was frequently practiced on tricycles, but we found very few references to rear-steered bicycles. Patents and illustrations, of course, are not evidence as to whether imagined designs ever worked in practice.

- Rankine [91](1869) said, as noted in R. Sharp & Limebeer [92], "A bicycle, then, with the steering wheel behind, may possibly be balanced by a very skillful rider as a feat of dexterity; but it is not suited for ordinary use in practice."
- A. Sharp [59] (1896) describes several rear-steering bicycle designs and then says "There have been very few rear-steering bicycles made, though their only evident disadvantage is that in turning aside to avoid an obstacle, the rear-wheel *may* foul, though the front-wheel has already cleared. Nearly all successful types of bicycles have been front-steerers." Sharp

also said [93] (1899) “Rear steering bicycles have never been popular,” without indicating what evolutionary forces selected out the rear-steerers.

- Lee Laiterman [94] (1977) developed a rear-steering bicycle for a Bachelor’s Thesis at M.I.T., and found that it required excessive concentration and control effort to balance.
- Schwarz [95] (1979). In the 1970’s the United States National Highway Safety Administration sponsored research into a rear-steering motorcycle expected to have safety benefits in rapid obstacle avoidance. The preliminary analysis of Schwarz concluded that a rear-steering design could never be stable, saying a large positive eigenvalue ‘is inherent in the rear steered configuration’ of a motorcycle. A machine was subsequently built, but could never be balanced upright for more than a couple of seconds. This led to widespread criticism of government wastefulness.
- Craig Cornelius [96] (1990) built several rideable rear-steering bicycles. Cornelius believed that rear-steering designs are inherently unstable, in contrast with the auto-stability of a front-steering bicycle.
- Åström *et al.* [97] (2005) present experiments and analysis that indicate that considerable difficulty in balance of rear-wheel steering is to be expected.
- In 2007 and 2008 Curan Wright [98] rode his nearly-conventional bicycle backwards for thousands of miles.

2.5 Technical papers addressing the factors required for self-stability

Numerical results showing the impact of gyroscopic or trail terms on the self-stability of a given design

Various investigators, justifiably put off by the complexity of the algebraic self-stability conditions for a general bicycle, performed numerical explorations as an alternative. Parameter studies generally increased or decreased quantities such as trail, wheel polar inertia, or steer axis tilt. A typical outcome was to document changes in the speed range for self-stability as one single parameter was varied. Some vocabulary for readers of this literature: for a conventional bicycle or motorcycle, the lower end of the stable velocity interval is called the ‘weave speed’, and the upper end is called ‘the capsized speed’, after the particular dynamic modes that become unstable at these speeds.

Demonstrations of the dependence of eigenvalues on trail, wheel moment of inertia, and many other design parameters have been carried out by, among others, Herfkens [38] (1949), Döhring [40] (1953), R. N. Collins [89] (1963), R. Sharp [99] (1971), Roland & Massing [100] (1971), Weir [101] (1972), Singh & Goel [102] (1975), Zellner & Weir [103] (1979), Dikarev, Dikareva and Fufaev [88] (1981), Franke *et al.* [104] (1990), R. W. van Zanten [105] (2000), Åström *et al.* [97] (2005), and V. Cossalter [90] (2006).

Experimental results showing the impact of gyroscopic or trail terms on the self-stability of a given design

Experimental investigations of how bicycle design variations affect self-stability or open-loop eigenvalues are extremely rare. We found just two papers, and neither varied wheel inertia or

trail. Stevens [106] (2009) investigated only steer axis tilt of a given front assembly with no handlebars. Weir & Zellner [107] (1979) placed extra masses at various locations on the frame or front assembly.

Analytical results about factors related to self-stability, with at least some generality

Few theoretical analyses have progressed beyond derivation of the equations of motion and the description of the eigenvalue structure for standard bicycles. Those that have looked into necessary or sufficient factors for self-stability typically used highly, perhaps overly, simplified bicycle models, and/or ended up incorrectly deriving the equations of motion for roll and steer degrees of freedom. A review of papers that use equations of motion of any kind to investigate bicycle lateral balance is given in MPRS [6]

We have not thoroughly reviewed the full contents of Boussinesq [108, 109] (1899) or Carvalho [31] so are not certain of all the claims therein. We might infer from the later investigations by Klein & Sommerfeld [2] (1910), which were based on Carvalho, that Carvalho did not make any general qualitative statements.

Klein & Sommerfeld appeared to conclude that a spinning front wheel was essential for the self-stability of any design. However, as discussed in detail in Chapter 3, they actually came to this conclusion only for the specific bicycles studied by Carvalho and Whipple.

The only other references we know with explicit analytical demonstrations of parameter requirements were simplified-model derivations by Synge, Kane, and Neĭmark & Fufaev, who were already cited in the section on steer axis tilt.

2.6 Modern conventional ideas about the main factors providing self-stability

We close with a few up-to-date references indicating conventional views of how trail and wheel inertia affect self-stability. The aim is to show that the historically documented ideas remain very much in the mainstream. Anyone reviewing popular magazines or books about bicycles (or motorcycles) will see them repeated almost monthly. Because virtually every enthusiast magazine, book or blog repeats the same points, there is no way to acknowledge them all.

In addition to those publications treating bicycle and motorcycle design – some aimed at specialists, and others at interested laypersons – similar material occasionally appears in scientifically oriented popular books and articles (e.g., in textbooks for elementary physics courses).

The most important modern article is the paper by David (D.E.H.) Jones [4] (1970), which is discussed in detail in Chapter 5. Our sense is that the significance Jones attached to trail has shaped how many people think today about bicycle and motorcycle phenomena, see Chapter 5. Here are some other references that indicate the modern consensus.

- Robinson [110] (1994) ties reduced trail to lower steer torque and reduced stability, with the opposite when trail is increased.
- Zinn [111] (1998) wrote about gyroscopic self-stabilization: “The smaller wheel is less stable. It has less angular momentum and thus provides the bike with less stability from gyroscopic action.”; “a bicycle can stay upright without a rider”; and “The gyroscopic effect of spinning wheels certainly adds to a bike’s stability.” Zinn also linked trail to stability: “The way to increase stability is to increase trail.” and “But clearly, the negative fork trail produced by increasing rake so dramatically . . . eliminated the bicycle’s self-righting capabilities. . .”

Zinn notes Jones did not find it credible for the gyro effects of a light front wheel to provide stability to a bicycle with a heavy no-hands rider. Further, Zinn notes that Jones showed that no-hands riding was possible without gyro torques, but not without trail. Therefore, to Zinn, gyroscopic effects are not practically relevant for bicycles. Of course, no-hands rideability does not speak directly to self-stability, because the no-hands rider plays an active, skilled role.

- Karnopp [43] (2004) states that the trail provides a self-centering tendency on bicycles and motorcycles, with a dynamic behavior similar to vertical-axis casters on carts.
- Cossalter [90] (2006) states “The value of the trail is most important for the stability of the motorcycle ...”, “If the value of the trail were negative [it would] ... seriously compromis[e] the motorcycle’s equilibrium”.
- Foale [87] (2006) notes two aspects of the effect of trail: “The motorcycle ... was ... a single-track vehicle in which the use of an inclined steering head was a convenient way to provide the front-wheel trail necessary for automatic straight-line stability.” And “If the tyre contact patch was in front of the steering axis (negative trail), then the torque generated would reinforce the original disturbance and so make the machine directionally unstable.”
- Parks & Thede [112] (2010) state that negative trail is dynamically unstable.

In addition to these are hundreds or perhaps thousands of others, mostly semi- or non-technical. In fact, we do not know of any articles addressing lateral self-stability that do not make at least one of the following claims:

- Gyroscopic terms are helpful for bicycle self-stability.
- Trail is helpful for bicycle self-stability.
- Bicycles depend on rider control for stability.

While each of these is perfectly true in proper context, and many authors do not make unambiguously incorrect statements, the statements above encourage opinions which are also common, but which are not true, such as:

- Gyroscopic terms are necessary for bicycle self-stability,
- Trail is necessary for bicycle self-stability,
- A gyroscopic wheel together with trail is sufficient for bicycle self-stability, and
- Bicycles cannot be self-stable.

We close this chapter by drawing attention to the little-known work of A.J.R. (Tony) Doyle [9, 84] (1987). While his research was about rider hands-on control, it shows sensitivity to the issues of self-stability. Doyle’s purpose was to tease out the rider’s contribution to stability, so he aimed to eliminate all contributions to self-stability. He did his utmost to remove all coupling of roll to steer, by eliminating trail, steer axis tilt, offset of front assembly mass center from steer axis, and so on. While superficially along the lines of Jones’s unrideable bikes (URBs), Doyle’s destabilized bike attempted a much more precise task: to eliminate all self-stabilizing influences and rider torso-motion influences so the rider’s hands alone affected the steer angle. We believe he did this approximately correctly.

Chapter 3

COMMENTS ON KLEIN & SOMMERFELD 1910 (K&S)

Starting in the late 1800's mathematician Felix Klein and physicist Arnold Sommerfeld (K&S) wrote a four-part treatise *Über die Theorie des Kreisels (On the Theory of Gyroscopes)* [2]. In an 1899 Nature review [32] of an early volume, Greenhill wrote: “the next century will have its work cut out for the mathematical treatment of the ... dynamics of the bicycle.” A decade later, perhaps in response to this challenge, K&S explored bicycle self-stability in a twenty-two page chapter (Section 8) of the fourth volume, starting on page 863. As stated in the preface of part IV, mathematician Fritz Noether [brother of Emmy Noether] provided the main ideas in Section 8.

The chapter derives the equations of motion of the Carvallo model, discusses the roles of a few parameters, and includes lengthy and clear explanations of various features of bicycle stability.

K&S wanted to know “to what extent gyroscopic effects are important in stability,” by which they clearly meant ‘self-stability’. In more detail they wrote: “It would be interesting to investigate to what degree the self-stabilization of the bicycle with an unmoving rider is possible, and how important gyroscopic actions are in that case. The process of stabilization is then one such that Rankine’s steer-controlling actions are partly provided by gyroscopic effects ...”. They summarize,

“Here, we are interested in the contribution of the gyroscopic effects to the results mentioned above [that bicycles can be self-stable]. We shall show, what has not been pursued by [Whipple and Carvallo], that by leaving out the gyroscopic effects the region of full stability would disappear; therefore that the gyroscopic effects, despite their smallness, are indispensable for the self-stability.” — (Klein & Sommerfeld [2], 1910)

“Uns interessiert hier der Beitrag der Kreiselwirkungen zu den erwähnten Resultaten. Wir werden zeigen, was bei den genannten Autoren nicht verfolgt ist, daß bei Fortfall der Kreiselwirkungen das Gebiet der vollständigen Stabilität verschwinden würde, daß also die Kreiselwirkungen trotz ihrer Kleinheit für die selbständige Stabilisierung unentbehrlich sind.” (original German, page 866)

Did they mean this assertion, that gyroscopic action was necessary for self-stability, to apply to all bicycles, or just to the specific models analyzed by the main authors they cited, namely Whipple and Carvallo? First we review the chapter generally, then we discuss this question in detail.

3.1 The chapter overall

In the bicycle dynamics literature this Chapter is unusual in a few regards: 1) the equations of motion are essentially correct (see the supplementary appendix of MPRS [6]; 2) they explicitly

confirm agreement with other works (Whipple and Carvallo); and 3) up to our present work, their treatment is perhaps the only analytically-based published investigation that explores the conditions and causes of self-stability.

K&S affirm Rankine's [91] assertion, that balance is accomplished by steering toward a fall. They note the potential role of small body movements by the rider. And they correctly dismiss the direct contribution of gyroscopic torques to righting a bicycle, pointing out two things: 1) if the steering angle is fixed, there is *no* (zero) gyroscopic stabilization because the gyroscopic torque from leaning is about the yaw (rather than the lean) axis; 2) because the wheel masses are relatively small, any gyroscopic torques from the spinning wheels, even if they were about the correct axis, are too small to correct the tipping moment of a falling bicycle.

Rather, according to K&S, leaning causes (through the gyroscopic torque) steering, which in turn causes the righting accelerations: "The proper stabilizing force, which overwhelms the force of gravity, is the centrifugal force, and the gyroscopic action plays the role of a trigger."

K&S use the analyses of Carvallo [31] and Whipple [3] in the following three ways. First, they adopt the slightly simplified bicycle model of Carvallo, in which the front assembly mass resides solely in the front wheel. (In 1953 Döhning [40] generalizes the K&S calculations to allow for more general front-assembly mass distributions.) Next, they note agreement between their equations and the equations of both Carvallo and Whipple. Lastly, they make use of the numerical examples of both Carvallo and Whipple, with respect to the ranges of speed where self-stability could be possible.

There is no indication that K&S performed any numerical evaluations themselves. Rather they use such wording as "Whipple finds", "We don't want to do those calculations but refer to those of Whipple", "The calculations of Carvallo for an older bicycle give qualitatively the same results". K&S's calculations are all analytical.

3.2 Equations of motion errors

First, we review the relatively minor errors in the K&S Equations of Motion. Table 3.1 on page 24 here shows the names of the parameters used by K&S and their relation to the parameters of MPRS [6]. The K&S linearized equations of motion are essentially correct, apart from a few typos:

- On page 875, Equation (9), third line, first term, $h_2 \sin \sigma$ should read s_2 , which is equal to $h_2 \sin \sigma + r \cos \sigma$.
- On page 878, in element (2,2) of the determinant, s should read s_2 .
- On page 880, first displayed equation, second line, second \cos should read $\cos \sigma$.
- On page 881, Equation (15), $\cos \sigma$ should be inserted before the final round bracket.

3.3 "Gyroscopic effects ... are indispensable for self-stability"

Then we turn to the central question of K&S: Are gyroscopic effects needed for bicycle self-stability? The paragraph translated at the beginning of this chapter (from page 866) has troubled and misled later investigators of bicycle dynamics. It is written informally so the intended meaning is difficult to pin down.

One interpretation is that K&S meant this to apply only to the two numeric example bicycles from Whipple and Carvallo. Here, we have checked, their conclusions are correct. Those two bicycles are not stable if the gyro terms are eliminated. Another interpretation is that they meant this to apply to all possible bicycles. In an attempt to resolve the authors' intentions, we examine their stability calculations and arguments.

Stability calculation and errors: The characteristic equation is the determinant shown on page 878. This equation has the structure, as shown on page 879,

$$\alpha\lambda^4 + \beta u\lambda^3 + (\gamma_1 + \gamma_2 u^2)\lambda^2 + (\delta_1 u + \delta_2 u^3)\lambda + (\epsilon_1 + \epsilon_2 u^2) = 0, \quad (3.1)$$

where λ is an eigenvalue, u is the forward speed and $\alpha, \beta, \gamma_1, \gamma_2, \delta_1, \delta_2, \epsilon_1$ and ϵ_2 are constants depending on the parameters.

As stated precisely by K&S, the necessary and sufficient conditions for self-stability are

- all the coefficients of the powers of λ must have the same sign. Since α is always positive (see Chapter 4 in Kooijman [1]) this means that all other coefficients must be positive
- the polynomial's Routh-Hurwitz discriminant, X , must also be positive. (See the Supporting Online Material of Kooijman *et al.* [1] for the definition of X .)

K&S present Whipple's results, that for the bicycle parameters investigated by him, the quadratic coefficient (γ polynomial) becomes positive above a calculable speed u_1 , and the linear coefficient (δ polynomial) becomes positive above a somewhat higher calculable speed u_2 . Lastly the constant coefficient (ϵ polynomial) changes from positive to negative above an even higher speed u_3 . It is therefore clear that this bicycle will have a range of stable speeds if the Routh-Hurwitz discriminant X becomes positive at some speed between u_2 and u_3 .

Their strategy is to investigate the linear coefficient giving rise to u_2 , namely $(\delta_1 u + \delta_2 u^3)$, which involves gyro terms. The explicit expression for this polynomial, numbered (13) on page 880, is derived from the correct unnumbered displayed expression on the same page. Their expression reads

$$(13) \quad \begin{aligned} & -gMh \cos \sigma (c_2 A_v + c_1 B_v) \frac{u}{l} \\ & + gB_{hv} \left(-M_1 h_1 \sin \sigma + M_2 r \frac{c_1}{l} \cos \sigma \right) u \\ & -gM_1 h_1 M_2 h_2 l \sin \sigma \cdot u - gM_1 M_2 h_1 c_1 r \cos \sigma \cdot u \\ & + \frac{c_2 + c_1}{l} \cos \sigma N [2Nu + Mhu^3]. \end{aligned}$$

The first three lines are proportional to the forward speed, u , and collectively form the quantity $\delta_1 u$. The terms on the fourth line contain N , the spin angular momentum of each wheel, which is also proportional to u . Thus terms on the last line are proportional to u^3 forming the quantity $\delta_2 u^3$, and will dominate the entire expression for sufficiently large speed. For Whipple's bicycle δ_1 is negative, and δ_2 is positive, which allowed Whipple to calculate the speed u_2 at which the entire expression becomes positive.

K&S considered the result of suppressing the gyroscopic contributions. In that non-gyroscopic case, δ_2 disappears, and stability will not even be possible unless the δ_1 term is positive.

K&S note that both Whipple and Carvallo calculated δ_1 to be negative, therefore precluding self-stability if gyroscopic effects are absent for their particular bicycles. Did K&S believe that δ_1 could never be positive?

We note first that K&S made an algebra mistake. The six monomial coefficients of u are products of quantities that are positive in a conventional bicycle design. Thus, the signs of their contribution are given by the + or - preceding each one. However in producing expression (13) from prior correct expressions, K&S made two sign errors. The correct expression should actually

read

$$\begin{aligned}
 & -gMh \cos \sigma (c_2 A_v + c_1 B_v) \frac{u}{l} \\
 & + gB_{hv} \left(\underbrace{+}_{\uparrow} M_1 h_1 \sin \sigma + M_2 r \frac{c_1}{l} \cos \sigma \right) u \\
 (13) \quad & -gM_1 h_1 M_2 h_2 l \sin \sigma \cdot u \quad + \underbrace{gM_1 M_2 h_1 c_1 r \cos \sigma \cdot u}_{\uparrow} \\
 & + \frac{c_2 + c_1}{l} \cos \sigma N [2Nu + Mhu^2].
 \end{aligned}$$

This error did not simply arise in typesetting, because it was incorporated into the reasoning of the authors, who write about the magnitude of the *single* positive term.

To examine the significance of these errors, we evaluated the individual terms numerically, truncating the bicycle parameters given by Whipple to fit the simpler bicycle model of K&S (putting the front assembly mass completely in the front wheel). Then the quantity (13) with the two sign errors marked becomes

$$\begin{aligned}
 & -(89.9967 + 489.7782)u \\
 (13) \quad & + \left(\underbrace{-}_{\uparrow} 67.8900 + 242.4644 \right) u \\
 & -271.5602u \quad - \underbrace{8.4863u}_{\uparrow} \\
 & + 50.5443u^3.
 \end{aligned}$$

K&S also report Whipple’s value (rounded) for the velocity $u_1 = 12$ km/h at which this expression becomes zero. Using Whipple’s parameters in the corrected expression we find $u_1 = 11.7$ km/h which is in accordance with the result that Whipple reports [3] (p.342 $V_4 = 11.8$ km/h). If K&S had calculated using their own erroneous expression, the inconsistent result $u_1 = 13.3$ km/h would have revealed their mistake.

With the corrected expression there are three positive terms rather than one. And it would be harder to argue that all conventional bicycles are likely to have a negative coefficient of u . To prove the generality of the need for gyroscopic terms, K&S would have had to prove that their positive term was always small, in comparison to the negative terms, for all stable bicycle designs. Without proof they write: “Of those terms proportional to u , the negatives outweigh the positive, because the positive term contains the small factors c_1 [trail] and r [distance of rear CM in front of rear contact].” This argument is not general, but depends on special parameter values; those “small” factors need not be small in all bikes. And some other relatively small parameters in the Whipple-based expression, such as steer axis tilt, can even change the sign of other terms in that expression. Thus the K&S c_1, r comment can only be interpreted as an *ad hoc* explanation of why for the particular Whipple bicycle, and perhaps others similar to it, the positive term does not outweigh the negatives. The phrase “when bicycle construction is suitable,” indicates that K&S knew they were not presenting a universal result. They give their conclusions about Whipple’s calculation this way:

“Die von Whipple gefundene Stabilität des Fahrrads für die Geschwindigkeiten von 16–20 km/h ist daher nur durch die Kreiselwirkungen der rotierenden Räder ermöglicht.“

Which we translate as:

“The stability of the bicycle found by Whipple for the speeds from 16–20 km/h is therefore only made possible through the gyroscopic effects of the wheels.”

Despite their calculation error, this conclusion is correct for the specific bicycle examined by Whipple.

We suspect that K&S hoped for some kind of generality in their conclusion about the need for gyro effects to get bicycle self-stability. But their reasoning is limited to thinking about bicycles fairly similar to Whipple's. Whether it was wishful thinking, or just casual writing, K&S's general statement "Gyroscopic effects . . . are indispensable for self-stability" is not supported in their writing, and is also not correct in general.

3.4 Other features of K&S

Here are some other interesting features of the K&S treatment of bicycle stability.

- They discuss, 60 years before Jones, the steer torque developed in a leaned bicycle due to gravity — what we call in Chapter 5 the *Jones Static Couple*. But they did so only to dismiss it, arguing that phase delay would not permit that gravity torque to stabilize a bicycle. Also as discussed in Chapter 5 here, the Jones Static Couple is inadequate to characterize the total steering torque arising from lean of an unsupported bicycle beginning to fall. A dynamic contribution of comparable magnitude is also needed, which both Jones and K&S ignore.
- If we interpret correctly, K&S also mention uncritically the supposed castering effect of trail, citing Bourlet. They do qualify their page 867 statement by saying "if the rear frame is held upright", but don't recognize that when the bicycle is rolling freely the potential self-centering from caster is more subtle.
- Although they show presence of gyroscopic terms in the E (ϵ polynomial) coefficient, they don't notice that reducing (or eliminating) the gyroscopic effects would increase the upper instability speed.

Table 3.1: Corresponding parameters and variables used by Klein & Sommerfeld [2] (1910) (K&S) and Meijaard *et al.* [6] (2007) (MPRS)

description	K&S [2]	MPRS [6]
<i>parameters</i>		
Mass of the front wheel	M_1	m_F
Mass of the rear frame assembly	M_2	$m_R + m_B$
Total mass	M	$m_R + m_B + m_H + m_F$
Radius of both wheels	h_1	$r_R = r_F$
Height of the center of mass of the rear frame assembly above the road	h_2	$(m_R r_R - m_B z_B)/(m_R + m_B)$
Height of the center of mass of the total system above the road	h	$(m_R r_R - m_B z_B - m_H z_H + m_F r_F)/(m_R + m_B + m_H + m_F)$
Forward distance of the center of mass of the rear frame assembly from the rear wheel contact point	r	$m_B x_B/(m_R + m_B)$
Moment of inertia of the front wheel about a vertical axis through its center	A_v	I_{Fzz}
Moment of inertia of the front wheel about a horizontal axis through the contact point	A_h	$I_{Fxx} + m_F r_F^2$
Moment of inertia of the rear frame assembly about a vertical axis through the rear wheel contact point	B_v	$I_{Bzz} + I_{Rzz} + m_B x_B^2$
Moment of inertia of the rear frame assembly about a horizontal axis through the rear wheel contact point	B_h	$I_{Rxx} + m_R r_R^2 + I_{Bxx} + m_B z_B^2$
Product of inertia of the rear frame assembly with respect to the rear wheel contact point	B_{hv}	$I_{Bxz} - m_B x_B z_B$
Steering head angle	σ	λ
Trail	c_1	c
Wheel base	l	w
Wheel base + trail	c_2	$w + c$
Spin angular momentum of both wheels	N	$I_{Ryy} v/r_R = I_{Fyy} v/r_F$
Gravity acceleration	g	g
<i>variables</i>		
Forward speed	u	v
Lean angle rear frame	ϑ_2	ϕ
Lean angle front wheel	ϑ_1	$\phi + \delta \sin \lambda$
Yaw angle rear frame	ϕ_2	$-\psi$
Yaw angle front wheel	ϕ_1	$-\psi - \delta \cos \lambda$
Steering angle	γ	$-\delta$
Vertical component of the steering angle	ψ	$-\delta \cos \lambda$

Chapter 4

COMMENTS ON WHIPPLE 1899

This chapter discusses Whipple's [3] 1899 paper "The stability of the motion of a bicycle". Whipple was 23 years old when he wrote this paper as an entry for the annual Cambridge University Smith Prize Competition, which he did not win. More details of his life and work are in the boxed item on p. 39 of R. Sharp & Limebeer [92]. Some aspects of Whipple's paper are reviewed in MPRS [6] and the supplementary material therein.

Because we were especially interested in the gyroscopic conclusions of K&S and because those conclusions rest in part on information from Whipple, we looked further into Whipple. Whipple's paper contains more printing errors than real errors. His non-linear equations are incorrect (because of a faulty expression of the front-contact vertical constraint), but that is not of central interest for our investigation of the essentially correct linearized equations. One of the boundaries which he calculates for the stable speed range for his example bicycle, the weave speed, has a minor calculation error

Here we provide a catalogue of issues related to Whipple. Except for someone planning to use Whipple's equations, as published by Whipple, it is not likely to hold much of interest. At the end of this chapter is a table of the bicycle parameters used by Whipple in his numerical calculations, using the MPRS notation.

p.313: No 2, trivial typo: 'intructions' → 'instructions'. 'McGaw' is G. T. McCaw, "On the steering of the bicycle", *The Engineer*, Vol. 86, p. 557 (Dec. 9, 1898).

p.314: The distance p' is not indicated in Fig. 2, but it appears to be minus the distance $O'Q'$. Fig. 3 is puzzling. Apparently, it is a top view, with V pointing towards us. B_2 is the forward direction, or x -direction. The H -system is common to the front frame and back-frame, but is a little awkward to use. $H_1M_2B_3$ is more convenient, with H_1 along the steering head hinge, M_2 in the direction of OQ and B_3 in the direction of the rear wheel axle. The H -system is found from this system by a rotation with an angle ϕ about H_1 . The angle θ is the actual angle the steer axis makes with the vertical. The angle ϕ is undefined for a vertical steer axis, which makes the choice of coordinates, ϕ and θ , inconvenient for a general configuration.

p.315: The four goniometric formulas can be derived with the help of rotation matrices and are correct. However, Eqn. I is not correct: it should read $(p \sin \eta + a) \cos \psi - (p' \sin \eta' + a') \cos \psi' = q \cos \theta$. If you take the time derivative, you get $\cos \theta$ times the first plus $\sin \theta$ times the second of Eqn. II. However, if you linearize, as on p. 320, the linearized result is the same as for the incorrect equation.

p.315: There are two typos in Eqn. II: a square bracket is missing at the end of the second line; a minus sign should be added after the left square bracket in the sixth line.

p.316: In Eqn. III, θ_2 , there should be θ_3 in the last term on the fourth line. In Eqn. IV, the second a in the third line should not be there. Unfortunately, Whipple uses g for the center of gravity

as well as for the acceleration of gravity. Similarly confusing is that B_3 is both an axis and a moment of inertia. Last line should read: force $Q_1Q_2Q_3$, not $F_1F_2F_3$.

p.318: On the 4th line, H_1 is meant, we presume. On the 4th and 5th line of displayed equation, the expressions between parentheses must be interchanged. On the second line before Eqn. VII, VI.(3) should read V.(3). On second line of Eqn. VII, remove subscript 1 from γ ; on the last line, there should be a δ before W (compare with Eqn. VIII). On the last line of the page, there should be an a between the (and \cos .

p.319: In Eqn. VIII, remove the subscript $_1$ from γ . In Eqn. IX, some terms quadratic in the angular velocities are missing in the right-hand side: $Ma(\theta_2 \cos \eta - \theta_1 \sin \eta)(c_1\theta_1 + c_2\theta_2)$. These terms disappear in the linearization. Furthermore, in Eqn. VII and IX is some confusion in the order of the indices in the inertia tensors Γ and E ; these tensors are symmetric, of course, so the order confusion does not matter.

p.320: On the 3rd line from below, III should read II.

p.321: On the 4th line, $\mu = TR$ should read $\mu = TS$ to be consistent with Fig. 2. In Eqn. VIII, the equation number is spoilt: the three dots should appear before the V. In Eqn. XI, the subscript 1 is missing in first P , and a prime is missing on $-P_1$. On the 4th line from below, there should be an extra dot over ϕ , or ϕ should be τ .

p.322: Apparently, it is assumed that the products of inertia E_{13} and E_{23} are zero, which is the case for symmetric frames. On line 8, there should be a prime on the second W . On the 3rd line from below, γ' should be κ' . On the 2nd line from below, the minus before W' should not be there.

p.323: On the 4th line, there should be an opening parenthesis. On the 5th line, the same expression with dashes, but with a minus sign before Λ (or $\Lambda' = -\Lambda$). Λ and also b are odd, so they have opposite signs with primed terms. Note, Whipple calls the primes (') 'dashes'. On the 11th line, the corresponding primed coefficient has $+\mu'\Pi$ (Π is odd).

p.323: In Eqn. XV, 2nd line, a dot over ϕ is missing; on the 3rd line, two dots over first ϕ' and one dot over second ϕ' are missing.

p.323: Here Whipple arrives at the linearized equations. Note that τ could be eliminated with the help of Eqn. XIII, yielding two second-order differential equations. For the tricycle, in section 23, this is done. These equations agree with MPRS in the following way. First use the lean angle ψ and the steering angle δ as independent coordinates. Then we have $\phi = \psi/(\sin \theta)$ and $\phi' = \delta + \psi/(\sin \theta)$. Use Eqn. XIII to eliminate τ and $\dot{\tau}$, then the first equation is identical to MPRS. To get the second equation of MPRS, multiply Eqn. XV by $(\mu\mu')/(b \cos \theta) = (f\mu)/(\cos \theta)$, and subtract $(f \sin \theta)/(\cos \theta)$ times the first from it. This gives the desired result, after some further substitutions. (f here is called μ in MPRS). The elegance of Whipple's paper is that it presents the equations in a form which is virtually symmetric with respect to the front and rear frame assemblies.

p.324: In Eqn. XVI, our (MPRS) steering angle is $\phi' - \phi$. Our (MPRS) lean angle is $\phi \sin \theta$.

p.326: In the determinant, in element (2,1), the subscript $_2$ should be a superscript (square) after λ ; in element (2,3), there should be no prime at first W ; the closing parenthesis is superfluous.

p.327: In Element (1,3) of the determinant, $\cos b$ should read $\cos \theta$. Two lines lower, the terms with $\Pi b \cos \theta$, and the closing parenthesis should not be there. In the next line, in the expression between brackets with μ 's, prime should be with the other μ .

p.329: In Eqn. XXVI, third line, the last term between parentheses should be $(h\gamma'/\mu' - h'\gamma/\mu)$ (the first μ should be μ').

p.330: Whipple makes some simplifications for the calculations of the moments of inertia: the mass of the wheels is not considered separately.

p.331: Whipple gets his calculations for the example bicycle basically correct but makes a calculation error in the expansion of the characteristic equation XXVIII. In the ζ^2 term he calculates $(3.73 - 4.6\varepsilon)\zeta^2$ which should be $(3.70 - 4.21\varepsilon)\zeta^2$. [Where Whipple uses the non-dimensional eigenvalue $\zeta = \lambda b/v$ and the inverse of the squared Froude number $\varepsilon = gb/v^2$ to characterize the forward speed, and b is “the unit of length, which may be though of as one metre”.] Consequently, three lines lower, ε_0 should be 0.878 instead of 0.81 (further on, the meaning of ε_0 and ε_1 is sometimes interchanged). The last equation becomes $\zeta^4 + 4.046\zeta^3 + 0.405\zeta + 1.400 = 0$, with of course different roots. This has implications for the calculation of the weave speed V_2 further on.

p.332 Continuing, this leads to the corrected fourth equation $-0.248\varepsilon^2 - 4.119\varepsilon + 2.401 = 0$, with root $\varepsilon_2 = 0.564$ instead of 0.505. This changes the calculated weave speed V_2 on p. 342. There should be a dot before the number 839.

p.333: Whipple discusses the idea of a very heavy rear wheel to increase, or eliminate, the capsize speed. We note that the bicycle approaches a rolling disk in that case. One could also get a rolling hoop by making the center of mass of the rear frame at the rear-wheel contact and making the moment of inertia about that point zero.

In an example of the mild capsize instability at $V = 10$ m/s, he states that perturbation grow by a factor e after 20 revolutions of the pedals. With the given formulas, we found the characteristic time to be about 3 seconds (about 5 pedal revolutions). Note that the trail in Whipple’s example is small: 5.5 cm. For the benchmark bicycle (MPRS) at the same speed, it takes about 6 seconds for the perturbations to increase by a factor of e .

p.334: In Eqn. XXIX, ω should be W ; the lower case v should be upper case V ; the lower case π should be upper case Π .

p.335: In Eqn. XXX, in first line, -7.5 should be $+7.5$. It is funny to see that Whipple rounds the same numbers differently in different places.

p.336: We get $\varepsilon = 0.8$ at $Z_2 = 0.24$ and $\varepsilon = 0.1$ at $Z_1 = -3.1$.

p.340: In Eqn. XXXIV, .86 should be 8.6; the coefficient before ε^2 should be about 1.45 instead of 2.4. Near the bottom of the page, the conversion from χ to $A\phi$ is not done correctly. The error is small, however.

p.341: In the first equation, .46 should be .36. This is a typo, because 0.36 is used further on.

p.342: Here Whipple slightly incorrectly reports for the weave speed of the example bicycle, which is derived from the Routh-Hurwitz discriminant X :

$$\varepsilon_2 = 0.505, \quad V_2 = 4.62 \text{ m./sec.} = 16.6 \text{ km./hr.} = 10.4 \text{ miles/hr.},$$

But with the corrected value for $\varepsilon_2 = 0.564$ from p.332 the correct values are:

$$\varepsilon_2 = 0.564, \quad V_2 = 4.38 \text{ m./sec.} = 15.75 \text{ km./hr.} = 9.79 \text{ miles/hr.}$$

p.347: In Eqn. XII, there should be a prime at C_3 .

Of all the foregoing, the comments of greatest relevance to the K&S discussion are those for page 342.

Parameter	Symbol	Value for benchmark
Wheel base	w	1.1 m
Trail	c	0.055 m
Steer axis tilt (90° – head angle)	λ_s	$\arctan(0.4) \approx 21.801^\circ$ ($90^\circ - 68.199^\circ$)
Gravity constant	g	9.81 N/kg
Forward speed	v	<i>various</i> m/s
Rear wheel R		
Radius	r_R	0.385 m
Mass	m_R	0 kg
Mass moments of inertia	(I_{Rxx}, I_{Ryy})	(0, 0.21175) kgm ²
Rear Body and frame assembly B		
Position center of mass	(x_B, z_B)	(0.275, -1.1) m
Mass	m_B	80 kg
Mass moments of inertia	$\begin{bmatrix} I_{Bxx} & 0 & I_{Bxz} \\ 0 & I_{Byy} & 0 \\ I_{Bxz} & 0 & I_{Bzz} \end{bmatrix}$	$\begin{bmatrix} 6.05 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6.05 \end{bmatrix}$ kgm ²
Front Handlebar and fork assembly H		
Position center of mass	(x_H, z_H)	(1.1, -0.44) m
Mass	m_H	2 kg
Mass moments of inertia	$\begin{bmatrix} I_{Hxx} & 0 & I_{Hxz} \\ 0 & I_{Hyy} & 0 \\ I_{Hxz} & 0 & I_{Hzz} \end{bmatrix}$	$\begin{bmatrix} 0.09075 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.09075 \end{bmatrix}$ kgm ²
Front wheel F		
Radius	r_F	0.385 m
Mass	m_F	0 kg
Mass moments of inertia	(I_{Fxx}, I_{Fyy})	(0, 0.21175) kgm ²

Table 4.1: Parameters of the Whipple example bicycle from [3].

Chapter 5

COMMENTS ON JONES 1970

In 1970, and then again in 2006, *Physics Today* published what is by far the best known paper on bicycle stability or self-stability: “*The Stability of the Bicycle*” by David E. H. Jones [4]. Jones memorably described his light-hearted quest as a search for an unrideable bicycle (URB). His anti-gyro bicycle, that he could ride no-hands only if it had trail, took hold of the imagination. So did his enumeration of two potential roles for trail: heading self-stability of the front wheel (like a caster); and a coupling of bicycle lean angle in a fall to the necessary corrective steering. As noted earlier, most of these ideas had been expressed repeatedly in the 100 years prior, but his article made the greatest impression. Most people interested in any technical way about bicycles know this paper, at least indirectly. One often hears, roughly “wasn’t there a paper about an unrideable bicycle (URB) in *Scientific American* in the 1970’s”. One is surprised if any paper about bicycle stability since 1970 does *not* refer to Jones.

Jones concluded that the caster trail c (the distance the front-wheel ground contact ‘trails’ behind the intersection of the steering axis with the ground) is important both for self-stability and for hands-off control. Here we review Jones’s experimental results and then review and critique his theory. Jones’s place in the history of ideas about bicycle stability is also discussed in Chapters 1 and 2.

5.1 Jones’s experiments

Jones modified conventional bicycles in various ways. Of primary interest here, he altered the spin angular momentum of the front assembly (the front wheel, fork, and handlebar) by adding a counter-spinning wheel; and he altered the trail, making it more or less positive, or even negative.

Self-stability. For the bicycles he modified, Jones found that both front-wheel spin momentum and positive trail were needed for riderless self-stability. If there was an added counter-spinning front wheel, or if the trail was negative, a pushed riderless bicycle would fall. On the other hand a conventional bicycle, or one with extra trail or with extra spin angular momentum, would balance on its own quite well.

Ability for a rider to balance with hands on the handlebars. By steering the handlebars appropriately, Jones was able to balance all bicycles, self-stable or not.

Ability for a rider to balance with hands off the handlebars. When riding no-hands, *i.e.*, controlling bicycle steering by upper-body motions in order to maintain balance, Jones had difficulty balancing a bicycle whose front-wheel gyroscopic term was canceled by an added, counter-spinning

wheel. And, he was unable to master no-hands balance of a bicycle with significantly negative trail.

Summary of Jones's experimental results. In Jones's experiments both positive trail and spin angular momentum were needed for self-stability. They were also needed for easy no-hands riding. There is no reason to doubt the trends Jones found in the neighborhood of a conventional bicycle design. Jones did not, however, offer any precise delineation of which bicycles would be self-stable, or which bicycles could be ridden no-hands.

5.2 Jones's theory

On the theoretical side Jones wanted to counter widely-quoted gyroscopic explanations of no-hands bicycle control (as presented, for example, in Sharp (1896) [59]), and uncontrolled no-hands bicycle self-stability (as presented, for example, in Klein & Sommerfeld (1910) [2]). His theory was motivated by his experimental observations that trail was an important factor in bicycle self-stability and no-hands rideability.

Jones did no dynamical analysis of the general type associated with Whipple or Carvallo and reviewed in MPRS [6]. Instead he focused on the intuitive idea (possibly drawn from Wilson-Jones [39]) that to be self-stable, a bicycle needs to turn toward the side to which it is falling. Thus he set out to calculate the rider-applied steering torque it would take to hold the handlebars in place when a bicycle is leaned.

Jones calculated handlebar torque via the derivative of system gravitational (potential) energy with respect to steer angle at fixed lean. Implicit in such a calculation is an assumption that *the bicycle is supported by horizontal forces at a fixed lean angle*. This bicycle is allowed to yaw so the front-contact support force is perfectly vertical.

Jones focused attention on the constant of proportionality between, say, rightward lean and leftward steer torque (needed to restrain the front wheel from turning right). Where the steer torque is the derivative of energy with respect to steer angle, differentiating this with respect to lean angle to find the constant of proportionality effectively yields the mixed second partial derivative of energy, which is an off-diagonal term in the gravitational stiffness matrix. Thus, Jones's calculated quantity is equivalent to the term $gK_{0\delta\phi}$ in the full dynamics equations of MPRS [6], whose notation we also use below. However, his simplified model did not incorporate the masses of the handlebars, front fork, and front wheel, which also affect the potential energy when the steering of a leaned bicycle is turned. In effect, he had assumed a negligible-mass wheel, fork, and handlebar, and his system is equivalent to a heavy point-mass rider attached to the frame of an otherwise massless bike.

Because Jones restricted his mass to that of the frame, or frame plus rigid rider, steer torque arises only when the *frame* would rise or fall due to steering of the leaned bicycle. For the small lean and steer angles of a linearized analysis, such rise or fall of the frame center of mass occurs only if the trail is non-zero. And, the desired sign of steer torque (a tendency to turn towards a lean) only occurs if the trail is positive, which allows the center of mass of a bike at a fixed rightward lean angle to reduce in height when the handlebars are turned right. For small lean and steer angles, a simple expression for what Jones was seeking is

$$(\text{weight on front contact}) \times (\text{trail}) \times (\cos \lambda_s).$$

That is because, with a massless or balanced front assembly, steer torque due to lean is entirely due to the vertical front contact force being out of the plane of the leaned bicycle, and acting on the trail-related moment arm of the front ground contact relative to the steering axis.

Note, even in the relatively simple world of linearized statics, the problem of a bicycle held at a fixed lean angle is confusing until sufficient information is specified. Is it to be held by a moment (of what orientation?) or a force (what location and orientation)? Jones's static theory (apart from his neglect of the front-assembly mass) seems to be well represented by the experiments of Lowell & McKell [81]. Lowell & McKell balanced (using gentle fingertip pressure against a wall) while sitting on a bicycle at rest, with upper body lean relative to the frame, and with the front wheel on a freely rotatable turntable. Lowell & McKell's experiment accomplished the Jones aim of achieving a vertical support force at the front contact of a bicycle held at a fixed lean angle, while bearing a vertical load. Lowell & McKell measured the handlebar torque needed to hold the steering straight as the bicycle frame lean angle was varied. They call this measured torque (which of course included the contributions from wheel and handlebar weight that Jones had neglected) the *Jones Couple*.

We will call the gravitational steer torque from lean angle and calculated with $gK_{0\delta\phi}$ the *Jones STATIC Couple*.

5.3 Critique of Jones's theory

Neglected masses

A simple critique is that Jones considered only bicycles for which the front assembly mass is negligible. This seems reasonable because the front assembly mass is typically much smaller than that of the rear frame and rider. However, when the trail is small the above described lean-induced steer torque arising from the front contact force can be less than the lean-induced torque due to

$$(\text{weight of front assembly}) \times (\text{offset of front assembly CM perpendicular to steer axis}).$$

Jones's theory is easily generalized to account for this neglect. The effect of any front-assembly weight imbalance can easily be incorporated into the Jones Static Couple by using either the full calculation $gK_{0\delta\phi}\phi_0$, or a Lowell & McKell style of experiment.

A more substantial issue is the following. Even for bicycles where Jones's calculation of the static steering torque due to lean is sufficiently accurate, or replaced by a fully correct calculation of $gK_{0\delta\phi}$, the Jones Static Couple has little meaning, as described below.

Neglected dynamics

Jones's overall concern, and ours as well, is dynamic stability.

Potential energy and dynamic stability. For systems near a stationary equilibrium there is a tight connection between the shape of the potential energy surface and dynamic self-stability: at a potential energy minimum a system is dynamically self-stable. But a bicycle is not such a system: steady motion forward is not a stationary equilibrium. In steady forward motion a bicycle's potential energy is not at a minimum but either at a saddle or a maximum, depending on mass and geometry details. Thus there is no easy translation from considerations concerning static torques and energy to consideration of dynamic self-stability.

The main flaw in Jones's reasoning is the idea that the steer torque of a moving bicycle which is instantaneously *at* a given lean angle is equivalent to that for a bicycle *held* at that lean angle. But this is not so. A bicycle in a state of leaning angular acceleration (i.e., accelerating its initial lean angle while the steering is still straight) has quite different ground contact forces than the vertical reaction force envisioned by Jones.

A simple example makes this most clear. If the bicycle mass distribution is well approximated by a point mass on the frame, the ground contact forces acting on the falling bicycle will be oriented precisely *along* the frame (no perpendicular component whatever), and will have no steering torque at all, no matter what the trail. By considering only potential energy of the frame, Jones’s theory is in effect a point mass theory.

In other words, if a bicycle is *held* at a rightward lean angle, the rider must indeed restrain the steering from turning toward the right, and $gK_{0\delta\phi}$ (calculated correctly by Jones in the point-mass-bicycle case) gives an accurate accounting of the necessary steer torque. But as soon as the bicycle is released to fall sideways, the ground forces immediately cease acting perpendicular to the ground, the steer torque changes (vanishes in the point-mass case), and Jones’ prediction is completely incorrect.

More general dynamics. For general mass distributions (i.e., not just a single point mass on the frame) there will indeed generally be a steering torque proportional to lean angle when the bicycle is free to fall. There are two sources for this dynamic torque

- The front support force deviates from the plane of the bicycle frame. This force acts on the ‘lever arm’ of the trail.
- The distribution of accelerating masses on the front assembly. That is, products of inertia couple the leaning to steering.

Neither of these mass-distribution effects, both resulting from angular acceleration, is captured by Jones’s calculations. And, again, for a point mass bicycle (Jones’s bicycle) these effects are zero.

Using the full equations of motion from MPRS [6] we can easily determine the dynamically generated steer torque due to falling. Assume a bicycle is at a given lean angle ϕ_0 , but as yet with no lean rate ($\dot{\phi}_0 = 0$). Assume also that the steering is fixed straight ahead ($\delta = 0$). Now we want to know the handlebar torque needed to prevent any steering just after the bicycle is released to fall.

With $\delta, \dot{\delta}$ and $\ddot{\delta}$ all zero, the first equation from MPRS [6] allows us to determine the initial roll acceleration $\ddot{\phi}$. Then the second equation allows us to calculate steer torque from ϕ_0 and the calculated $\ddot{\phi}$. The result is just two terms:

$$\text{Steer torque} = \underbrace{gK_{0\delta\phi}\phi_0}_{\text{Jones Static Couple}} + \underbrace{\left(-M_{\delta\phi}/M_{\phi\phi}\right)gK_{0\phi\phi}\phi_0}_{\text{Dynamic Correction}}$$

In general the dynamic correction can be positive or negative, large or small. In the case of a point mass bicycle, it turns out that that $K_{0\delta\phi}/M_{\delta\phi} = K_{0\phi\phi}/M_{\phi\phi}$ and the net resulting torque due to lean angle in a fall is exactly zero, as explained above. Mass on the front assembly that is high or low, ahead or behind the axis, can potentially alter the sign of this torque. Only in the special limit where $M_{\phi\phi}$ approaches infinity and totally dominates the mass matrix does the Jones Static Couple correctly predict the steer torque. This situation is approximately obtained if a long and heavy tight-rope balance bar is rigidly mounted perpendicular to the frame, increasing lean angle inertia by a factor of 10 or more.

Instead of calculating the torque to hold the steering fixed in a fall, we might ask what steer acceleration occurs in a fall if the steering is not held at all. It turns out that the answer is simply the previously calculated torque, times the always negative quantity

$$-M_{\phi\phi}/\det(\mathbf{M}).$$

So calculation of the torque to hold the steering straight when a bicycle is released, or calculation of the initial steering acceleration if the steering is free, give proportional answers — both predict steering towards a fall in the same situations.

An extra caveat. Even this dynamic calculation is suspect as an index for self-stability. Although turning towards a fall seems to be the key to bicycle self-stability, we have no evidence that this torque is either necessary or sufficient for self-stability in general.

No-hands control authority. While Jones's calculation seems to have nothing to say about self-stability of a riderless bicycle, it does seem useful for determining control authority in no-hands riding. Part of the steer torque due to frame lean created by rider bending could be approximated reasonably well by static calculations. The correct calculation would then be the Jones Static Couple based on frame lean angle. So, Jones is correct to add trail effects to the precessing gyroscope effects in explaining how no-hands body bending steers a bicycle.

Castering. Separate from Jones's notion that trail provides the needed coupling between leaning and steering, he also noted the importance of the castering effect, "The bicycle has only geometrical caster [trail] stability to provide its self-centering." But even that intuitive notion is not supported by the full dynamic analysis. As trail is reduced to zero and made negative, none of our investigations have shown the expected catastrophic tendency of the front wheel to reverse direction. Indeed, the fast castering mode (the most negative eigenvalue trace) remains about as stable as before. Why? Because in the full model the bicycle frame is not immovable, and the coupled equations have more subtle dynamics.

5.4 Summary concerning Jones

On the plus side:

- 1) Jones's experimental trends are not in doubt: In the neighborhood of a conventional bicycle design, both trail and spin angular momentum help the self-stability, and both help in no-hands riding.
- 2) Jones's theory approximately calculates an effect of trail that aids control authority for no-hands riding (adding to the gyro-precession control authority).

On the minus side:

- 3) Jones (like Wilson-Jones before him) believes that self-stability depends on a leaned bicycle steering towards a fall. Although this seems to be a trend, there is as yet no firm evidence that it is always necessary for self-stability. And it certainly is not sufficient.
- 4) Jones's calculation is essentially that of $gK_{0\delta\phi}$, but he neglects the mass of the front assembly. Even with a relatively light front basket, the contribution of front-assembly mass to $gK_{0\delta\phi}\phi_0$ can be significant, especially if trail is small.
- 5) Most importantly, Jones wrongly calculates the torque required to restrain the steering of a falling bicycle by using an incorrect statics calculation that assumes it is supported against falling. The correct dynamic calculation yields

$$\text{Steer torque} = gK_{0\delta\phi}\phi_0 + (-M_{\delta\phi}/M_{\phi\phi})gK_{0\phi\phi}\phi_0.$$

We describe this as the *Jones Static Couple*, plus a *Dynamic Correction*. And the dynamic correction is generally large (e.g. totally negates the Jones Static Couple in the case of a point mass bicycle).

Primarily because of item (5) above, we see no connection between Jones's theory and anything we know about bicycle self-stability, except that trail c plays a significant, if complicated, role in the equations of motion.

5.5 Relation to our present claims

The gyroscopic explanation of bicycle self-stability dominated 19th and early 20th century thinking. Although Jones was not the first to note this (see Chapters 1 and 2), and notwithstanding his questionable theory, Jones' experiments and thoughts firmly established the notion that trail can have a large effect in bicycle self-stability. The present Science paper can be interpreted as a third step in that process, showing that a myriad of other parameters can also have strong effects on self-stability. These include the mass distributions on the rear frame and the handlebar assembly as well as the rake angle. But if we had to boil it down to one variable, it would be the location of the front-assembly center of mass (for example, in relation to the system center of mass and the steer axis). K&S said "gyroscopes". Jones said "also trail". And, we add "other terms, interacting in complex ways, especially the location of the front-assembly mass."

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Finally, we relied heavily on *Google Books Advanced Search*. At least for users within the United States, it provides an unparalleled ability to perform full-text searches on innumerable books, magazines, and sometimes scientific journals. The majority of our citations were uncovered by date-restricted searches on words and phrases, such as *bicycle*, *velocipede*, *stability*, *unstable*, *balance*, *gyroscope*, *caster*, *trail*, *no hands*, or their foreign-language equivalents.

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Where these texts are available through Google Books we have given the relevant URL. This delivers a web page from which one can search for title, author, or content words. (If nothing else, a random word or number can deliver one or more document pages, from which one can scroll to the specific cited pages.) Unfortunately, many Google Books items do not seem to be viewable outside the U.S.. Also, some URLs occasionally undergo revision, so they are not all stable.

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